M/M/1 Queueing system with Customer Balking and Reneging

Ch. Swathi, V. Vasanta Kumar, S. Hanumantha Rao

Abstract— This paper deals with an M/M/1 queueing system with customer balking and reneging. Balking and reneging of the customers are assumed to occur due to non-availability of the server during vacation and breakdown periods. Steady state probabilities for both the single and multiple vacation scenarios are obtained by employing probability generating functions. We evaluate the explicit expressions for various performance measures of the queueing system.

Key words:— impatient customers, balking, reneging, vacation, server breakdowns.

1. INTRODUCTION

In real life, it can be observed that most queueing systems are with server vacations, breakdowns, delayed repairs and impatient customers. Some of the common examples include those in manufacturing systems, designing of local area communication networks and data communication networks. In general, the customers are impatient. In our fast life we often see that the customers anticipating service need to form a queue. Going through impatience due to the phenomenon of joining queue, the customers may not join the queue or even if they join, may quit from the queue without before served.

The primary aim of this paper is to explore the steady state behavior of the M/M/1 queue with impatient customers under single and multiple vacation policies. The customer impatience with vacation has become the essential feature for the queuing models which are analyzed by the many authors in the past.

For the queueing models with balking, reneging during different vacation states the reader may refer to [4 and 5] Dequan Yue, Wuyi Yue, Xiuju Li [1] considered queuing system with impatient customers and multiple vacations under two-phase service. He derived the Probability generating functions were derived for different states of the server focusing the number of customers present in the system. Besides, the closed-form expressions were derived for various performance measures. These measures included the mean system sizes for various states of the server, the average rate of balking, the average rate of reneging and the average rate of loss. Sherif I. Ammar [2] found the transient behavior of an M/M/1 queue with impatient customers and multiple vacations. He obtained the explicit expressions for the mean and variance of the system size in terms of the modified Bessel functions.

A recent study by S. Hanumantha Rao, V. Vasanta Kumar et.al [3] has focused on the impatient behavior of customers in a two-phase M/M/1 queuing system with server breakdown and delayed repair. They derived the probability generating function of the queue length distribution in steady state, the mean system size and the average rate of loss.

In this paper, we consider M/M/1 queuing system with customer balking and reneging during the server vacations or breakdown periods. The current research is presented as per the following sections: The second section presents the mathematical model. The model is considered having multiple vacation policy and server breakdown. The equilibrium analysis of the system states is performed in the third section. Using this, the probability generating functions of the steady state probabilities are obtained. In section 4, the closed form expressions for some performance measures are derived. These measures include mean system size, average rate of customers served and balked. We considered the system under the multiple vacations. In fifth and sixth we describe the mathematical model and derived the steady state probabilities for the system with breakdowns under single vacation policy and obtained the various closed form results respectively. In section seven and eight conclusions and references are given respectively.

2. MULTIPLE VACATION POLICY

In the current research M/M/1 queue is considered. Here the server takes multiple vacations. Whenever the server has no customers to serve, the server will enter vacation mode. After the vacation period it returns to the system to resume its services. If the queue line is headed up with the customers for service, it starts its busy period and continues the services until the queue line impoverish. On the other hand, if there are no customers in the queue line by the time the server returns from vacation, the server takes another vacation.

Model description

The assumptions and notations of the system model in the present research are presented below.

a) Customers will arrive into the queue individually with the Poisson arrival rate λ. Also, the service times are negative exponentially distributed with rate μ.

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b) If the queue line is not having any customers, the server will go on for vacation. Here, the vacation time is considered to be exponential random variable with a parameter γ.

c) It is considered that all the customers who arrive may not join throughout the conditions of server vacation, busy service, and breakdown times. It has been assumed that during vacation time, the probability of an arriving customer joining the queue is given by b_1 and balk with probability $(1-b_1)$; during busy time an arriving customer join the queue with probability b_2 and balk with probability $(1-b_2)$; during breakdown or repair time an arriving customer join the queue with probability b_3 and balk with probability $(1-b_3)$.

d) During the vacation period, customers become impatient. If a customer arrives to get service from the system and finds that the server is on vacation, customer will set off a timer 'T'. This value is exponentially distributed with parameter ξ with a condition of abandoning and never returning to the system in case the customer's service servicing is not completed before the expiration of time T.

e) During the service phase, server breakdown is possible at any moment, server breakdowns assume Poisson distribution with parameter α.

f) Server repair times assume exponentially distribution with the rate η.

In this model, the state of the server represented by K

$$K = \begin{cases} 0, & server\ is\ on\ vacation \\ 1 & serve\ is\ on\ working \\ 2 & breakdown\ and\ waiting\ for\ repair \end{cases}$$

and N be the total number of jobs in the system.

We define, $\pi_{k,n}$=Pr(K=k,N=m) ,(k=0,1,2;m=1,2,3,…)

$$\lambda b_1 \pi_{0,0} = \xi \pi_{0,1} + \mu \pi_{1,1} \tag{2.1}$$

$$(\lambda b_1 + n\xi + \gamma)\pi_{0,m} = \lambda b_1 \pi_{0,m-1} + (n+1)\xi \pi_{0,m+1} \ , m \geq 1. \tag{2.2}$$

$$(\lambda b_2 + \mu + \alpha)\pi_{1,1} = \mu \pi_{1,2} + \gamma \pi_{0,1} + \eta \pi_{2,1}. \tag{2.3}$$

$$(\lambda b_2 + \mu + \alpha)\pi_{1,m} = \lambda b_2 \pi_{1,m-1} + \mu \pi_{1,m+1} + \gamma \pi_{0,m} + \eta \pi_{2,m} \quad , m \geq 2. \tag{2.4}$$

$$(\lambda b_3 + \eta)\pi_{2,1} = \alpha \pi_{1,1} + \xi \pi_{2,2}. \tag{2.5}$$

$$(\lambda b_3 + \eta + (m-1)\xi)\pi_{2,m} = \lambda b_3 \pi_{2,m-1} + \alpha \pi_{1,m} + n\xi \pi_{2,m+1}, \ m \geq 2. \tag{2.6}$$

3. STEADY STATE ANALYSIS

We define the following generating functions to solve the steady state equations

$$G_0(u) = \sum_{m=0}^{\infty} \pi_{0,m} u^m, \quad G_1(u) = \sum_{m=1}^{\infty} \pi_{1,m} u^m \ , \ G_2(u) = \sum_{m=1}^{\infty} \pi_{2,m} u^m. \tag{3.1}$$

Multiply equation (2.2) by u^m and summing over m from 1 to ∞, use (2.1) and (3.1) and simplify. Then we have

$$\xi(1-u)G_0^1(u) - (\lambda b_1(1-u)+\gamma)G_0(u) = -(\mu\pi_{1,1} + \gamma\pi_{0,0}). \tag{3.2}$$

Let $A = (\mu \pi_{1,1} + \gamma \pi_{0,0})$

Then (3.2) can be written as

$$G_0^1(u) - \left(\frac{\lambda b_1}{\xi} + \frac{\gamma}{\xi(1-u)}\right)G_0(u) = -\frac{A}{\xi(1-u)}. \tag{3.3}$$

Multiplying this equation by $e^{-\frac{\lambda b_1}{\xi}u}(1-u)^{\frac{\gamma}{\xi}}$ and integrating from 0 to u, we get

$$e^{-\frac{\lambda b_1}{\xi}u}(1-u)^{\frac{\lambda}{\xi}}G_0(u)=-\frac{A}{\xi}\int_0^u e^{-\frac{\lambda b_1}{\xi}u}(1-y)^{\frac{\gamma}{\xi}-1}dy+c \tag{3.4}$$

Put u =0 in this equation, to get $G_0(0)=C.$

By substituting the value of C in (3.4), we get

$$G_0(u)=e^{\frac{\lambda b_1}{\xi}u}(1-u)^{\frac{\gamma}{\xi}}G_0(0)-\frac{A}{\xi}e^{\frac{\lambda b_1}{\xi}u}(1-u)^{-\frac{\gamma}{\xi}}\int_0^u e^{-\frac{\lambda b_1}{\xi}u}(1-y)^{\frac{\gamma}{\xi}-1}dy \tag{3.5}$$

Since $G_0(1)>0$ and $\underset{u\to 1}{Lt}(1-u)^{\frac{\gamma}{\xi}}=0$, we must have

$$G_0(0)=\frac{A}{\xi}\int_0^1(1-y)^{\frac{\gamma}{\xi}-1}e^{-\frac{\lambda b_1}{\xi}y}dy. \tag{3.6}$$

Let $K=\int_0^1(1-y)^{\frac{\gamma}{\xi}-1}e^{-\frac{\lambda b_1}{\xi}y}dy$ (3.7)

Then, $G_0(0)=\pi_{0,0}=\frac{A}{\xi}K=\left(\frac{\mu\pi_{1,1}+\gamma\pi_{0,0}}{\xi}\right)k$ (3.8)

$$\pi_{0,0}=\frac{(\mu\pi_{1,1})K}{(\xi-\gamma K)} \tag{3.9}$$

From equation (3.5), $G_0(u)=e^{\frac{\lambda b_1}{\xi}u}(1-u)^{-\frac{\gamma}{\xi}}G_0(0)\left[1-\frac{\int_0^u e^{-\frac{\lambda b_1}{\xi}y}(1-y)^{\frac{\gamma}{\xi}-1}dy}{\int_0^1 e^{-\frac{\lambda b_1}{\xi}y}(1-y)^{\frac{\gamma}{\xi}-1}dy}\right]$ The probability that the sever is on vacation is given by $\sum_{m=0}^{\infty}\pi_{0,1}=G_0(1)$

$$G_0(1)=e^{\frac{\lambda b_1}{\xi}}G_0(0)\underset{z\to 1}{Lt}\frac{\left[-\frac{\int_0^u e^{-\frac{\lambda b_1}{\xi}y}(1-y)^{\frac{\gamma}{\xi}-1}dy}{\int_0^1 e^{-\frac{\lambda b_1}{\xi}y}(1-y)^{\frac{\gamma}{\xi}-1}dy}\right]}{-\frac{\gamma}{\xi}(1-u)^{\frac{\gamma}{\xi}-1}}$$

$$G_0(1) = e^{\frac{\lambda b_1}{\xi}} G_0(0)\left[\frac{\xi e^{-\frac{\lambda b_1}{\xi}}}{\gamma} \frac{1}{K}\right] = \frac{\xi}{\gamma K} G_0(0) = \frac{\xi}{\gamma K}\pi_{0,0} \tag{3.10}$$

Use (3.8) in (3.10) to get

$$G_0(1) = \frac{\xi}{\gamma K}\pi_{0,0} = \frac{\xi}{\gamma K}\left(\frac{AK}{\xi}\right) = \frac{A}{\gamma} = \frac{\mu\pi_{1,1} + \gamma\pi_{0,0}}{\gamma} \tag{3.11}$$

We multiply equation (2.3) by u and equation (2.4) by u^m, summing over m from 1 to ∞. Then

$$\left[(\lambda b_2 u - \mu)(1-u) + \alpha u\right] G_1(u) = \gamma z\, G_0(u) + \eta\, u\, G_2(u) - Au. \tag{3.12}$$

Substitute u=1 in (3.12)

$$G_1(1) = \frac{\eta\, G_2(1)}{\alpha} \tag{3.13}$$

Differentiate (3.12) w.r.to ‘u’ and evaluating at u=1, then

$$G_1^{|}(1) = \frac{(\lambda b_2 - \mu - \alpha)}{\alpha} G_1(1) + \frac{\gamma}{\alpha} G_0^{|}(1) + \frac{\eta}{\alpha} G_2^{|}(1) + \frac{\eta}{\alpha} G_2(1) \tag{3.14}$$

We multiply equation (2.5) by u and equation (2.6) by u^m $(m \geq 2)$, summing over m from 1 to ∞, we get

$$\xi u(1-u) G_2^{|}(u) - \left[\lambda b_3 u(1-u) + \xi(1-u) + \eta u\right] G_2(u) = -\alpha u\, G_1(u) \tag{3.15}$$

Substitute $G_1(u)$ from (3.12) in (3.15), to get

$$G_2^{|}(u) - \left(\frac{\lambda b_3}{\xi} + \frac{1}{u} - \frac{\eta}{2\xi}\left(\frac{-2\lambda b_2 u + \mu + \alpha + \lambda b_2}{-\lambda b_2 u^2 + (\lambda b_2 + \mu + \alpha)u - \mu}\right) - \frac{\eta}{2\xi}\frac{(\lambda b_2 - \mu + \alpha)}{\left(u - \dfrac{\lambda b_2 + \mu + \alpha}{2\lambda b_2}\right)^2 - p^2}\right) G_2(u) =$$

$$\frac{\alpha\left(\mu\pi_{1,1} + \gamma\pi_{0,0} - \gamma\, G_0(u)\right)u}{\xi(1-u)\left(-\lambda b_2 u^2 + (\lambda b_2 + \mu + \alpha)u - \mu\right)} \quad , where\ p = \sqrt{\left(\frac{\lambda b_2 + \mu + \alpha}{2\lambda b_2}\right)^2 - \frac{\mu}{\lambda b_2}} \tag{3.16}$$

Integrating factor of this equation is

$$\exp\left\{-\frac{\lambda b_3}{\xi} u\left(\frac{1}{u}\left(-\lambda b_2 u^2 + (\lambda b_2 + \mu + \alpha)u - \mu\right)^{\frac{\eta}{2\xi}}\left(\frac{p + u - \theta}{p - u + \theta}\right)^{\frac{\eta(\alpha + \lambda b_2 - \mu)}{4\xi p}}\right)\right\} \text{ where } \theta = \frac{\lambda b_2 + \mu + \alpha}{2\lambda b_2}.$$

Multiplying this equation (3.15) by

$$e^{-\frac{\lambda b_3}{\xi} z} \frac{1}{u}\left(-\lambda b_2 u^2 + (\lambda b_2 + \mu + \alpha)u - \mu\right)^{\frac{\eta}{2\xi}}\left(\frac{p + u - \theta}{p - u + \theta}\right)^{\frac{\eta(\alpha + \lambda b_2 - \mu)}{4\xi p}}$$

on both sides and integrating, we get

$$G_2(u)=e^{\frac{\lambda b_3}{\xi}u}u\left(-\lambda b_2u^2+(\lambda b_2+\mu+\alpha)u-\mu\right)^{-\frac{\eta}{2\xi}}\left(\frac{p+u-\theta}{p-u+\theta}\right)^{\eta\frac{\alpha+\lambda b_2-\mu}{4\xi p}}$$

$$\int_0^u e^{-\frac{\lambda b_3}{\xi}y}\left[\left(-\lambda b_2y^2+(\lambda b_2+\mu+\alpha)y-\mu\right)^{\frac{\eta}{2\xi}}\left(\frac{p+u-\theta}{p-u+\theta}\right)^{\eta\frac{\alpha+\lambda b_2-\mu}{4\xi p}}\right]$$

$$\frac{\alpha\left(\mu P_{1,1}+\gamma P_{0,0}-\gamma G_0(y)\right)}{\xi(1-y)\left(-\lambda b_2u^2+(\lambda b_2+\mu+\alpha)y-\mu\right)} \tag{3.17}$$

$$Now,\mu\pi_{1,1}+\gamma\pi_{0,0}-\gamma G_0(y)=\gamma G_0(1)-\gamma G_0(y)$$

$$=\frac{\xi}{K}\pi_{0,0}-\gamma G_0(y)$$

$$=\frac{\xi}{K}\pi_{0,0}-\gamma e^{\frac{\lambda b_1}{\xi}y}\pi_{0.0}F(y)$$

Where $F(u)=\dfrac{\left[1-\dfrac{\int_0^u e^{-\frac{\lambda b_1}{\xi}y}(1-y)^{\frac{\gamma}{\xi}-1}dy}{\int_0^1 e^{-\frac{\lambda b_1}{\xi}y}(1-y)^{\frac{\gamma}{\xi}-1}dy}\right]}{(1-u)^{\frac{\gamma}{\xi}}}=\dfrac{\left[1-\dfrac{\int_0^z e^{-\frac{\lambda b_1}{\xi}y}(1-y)^{\frac{\gamma}{\xi}-1}dy}{K}\right]}{(1-u)^{\frac{\gamma}{\xi}}}$.

From (3.16)

$$G_2(u)=\frac{\alpha}{\xi}e^{\frac{\lambda b_3}{\xi}u}u\left(-\lambda b_2u^2+(\lambda b_2+\mu+\alpha)u-\mu\right)^{-\frac{\eta}{2\xi}}\left(\frac{p+u-\theta}{p-u+\theta}\right)^{\eta\frac{(\alpha+\lambda b_2-\mu)}{4\xi p}}H(u)\pi_{0,0} \tag{3.18}$$

Where

$$H(u)=\int_0^u e^{-\frac{\lambda b_3 y}{\xi}}\frac{\left(-\lambda b_2y^2+(\lambda b_2+\mu+\alpha)y-\mu\right)^{\frac{\eta}{2\xi}}}{y}\left(\frac{p+u-q}{p-u+q}\right)^{\eta\frac{(\alpha+\lambda b_2-\mu)}{4\xi p}}R(y)dy$$

With

$$R(u)=\frac{\left(\frac{\xi}{K}-\gamma e^{\frac{\lambda b_1}{\xi}u}F(u)\right)u}{(1-u)\left(-\lambda b_2u^2+(\lambda b_2+\mu+\alpha)u-\mu\right)}.$$

Put u=1 in (3.18)

$$G_2(1)=\frac{\alpha}{\xi}e^{\frac{\lambda b_3}{\xi}}(\alpha)^{-\frac{\eta}{2\xi}}\left(\frac{p-\theta+1}{p+\theta-1}\right)^{\eta\frac{(\alpha+\lambda b_2-\mu)}{4\xi p}}H(1)\pi_{0,0}. \tag{3.19}$$

Using the normalization conditions, we have
$G_0(1)+G_1(1)+G_2(1)=1$.
We obtain

$$P_{0,0}=\left[\frac{\xi}{\gamma K}+\left(1+\frac{\eta}{\alpha}\right)\frac{\alpha}{\xi}e^{\frac{\lambda b_3}{\xi}}(\alpha)^{-\frac{\eta}{2\xi}}\left(\frac{p-q+1}{p+q-1}\right)^{\eta\frac{(\alpha+\lambda b_2-\mu)}{4\xi p}}H(1)\right]^{-1} \quad (3.20)$$

Differentiate (3.2) w.r.to' *u*' and evaluating at u=1, we have

$$G_0'(1)=\frac{\lambda b_1}{\gamma+\xi}G_0(1) \quad (3.21)$$

Differentiate (3.15) w.r.to '*u*' and evaluating at u=1, we have

$$G_2'(1)=\frac{[\eta(\lambda b_2-\mu)+\alpha(\lambda b_3+\xi)]G_2(1)+\alpha\gamma G_0'(1)}{\alpha\xi} \quad (3.22)$$

4. PERFORMANCE MEASURES

i) In vacation period of server, the average number of jobs waiting in the queue

$$E(L_0)=G_0'(1)=\frac{\lambda b_1\xi}{(\xi+\gamma)\gamma K}\pi_{0,0} \quad (4.1)$$

ii) In busy period of server, the average number of jobs waiting in the queue

$$E(L_1)=G_1'(1)=\frac{(\lambda b_2-\mu-\alpha)}{\alpha}G_1(1)+\frac{\gamma}{\alpha}G_0'(1)+\frac{\eta}{\alpha}G_2'(1)+\frac{\eta}{\alpha}G_2(1) \quad (4.2)$$

iii) In breakdown period and waiting for repair, the average number of jobs waiting for repair in the queue

$$E(L_2)=G_2'(1)=\frac{[\eta(\lambda b_2-\mu)+\alpha(\lambda b_3+\xi)]G_2(1)+\alpha\gamma G_0'(1)}{\alpha\xi} \quad (4.3)$$

iv) The system size of the queue

$$E(L)=E(L_0)+E(L_1)+E(L_2) \quad (4.4)$$

v) By Little's formula mean waiting time for job in the system

$$E(W)=\frac{E(L)}{\lambda} \quad (4.5)$$

vi) The proportion of jobs served per unit time

$$P_s=\frac{\mu G_1(1)}{\lambda} \quad (4.6)$$

vii) The proportion of customers reneged per unit time

$R_m=\sum_1^\infty m\,\xi\,\pi_{0,m}$. (4.7)

viii) The Proportion of jobs balked per unit time

$$R_{balk}=\sum_{m=0}^{\infty}\lambda\left(1-b_0\right)\pi_{0,m}+\sum_{m=0}^{\infty}\lambda\left(1-b_1\right)\pi_{1,m}+\sum_{m=0}^{\infty}\lambda\left(1-b_2\right)\pi_{2,m}+\sum_{m=0}^{\infty}\lambda\left(1-b_3\right)\pi_{3,m} \quad (4.8)$$

5. SINGLE VACATION POLICY

We now consider the M/M/1 queue where the server takes only a single vacation at the end of a busy period.

The steady state difference equations are derived and presented below:

$$\left(\lambda b_1+\gamma\right)\pi_{0,0}=\xi\pi_{0,1}+\mu\pi_{1,1} \quad (5.1)$$

$$\left(\lambda b_1+n\xi+\gamma\right)\pi_{0,m}=\lambda b_1\pi_{0,m-1}+\left(m+1\right)\xi\pi_{0,m+1}\ ,\ m\geq 1 \quad (5.2)$$

$$\lambda b_2\pi_{1,0}=\gamma\pi_{0,0} \quad (5.3)$$

$$\left(\lambda b_2+\mu+\alpha\right)\pi_{1,m}=\lambda b_2\pi_{1,m-1}+\mu\pi_{1,m+1}+\gamma\pi_{0,m}+\eta\pi_{2,m}\ ,\ m\geq 1 \quad (5.4)$$

$$\left(\lambda b_3+\eta\right)\pi_{2,1}=\alpha\pi_{1,1}+\xi\pi_{2,2} \quad (5.5)$$

$$\left(\lambda b_3+\eta+\left(n-1\right)\xi\right)\pi_{2,m}=\lambda b_3\pi_{2,m-1}+\alpha\pi_{1,m}+n\,\xi\pi_{2,m+1}\ ,\ m\geq 2 \quad (5.6)$$

6. THE STEADY STATE ANALYSIS & RESULTS

The generating functions defined in the multiple vacations model are used to figure out the steady state equations

Multiply equation (5.2) by u^m, summing over m from 1 to ∞ and using (5.1), we have

$$\xi\left(1-u\right)G_0^1\left(u\right)-\left(\lambda b_1\left(1-u\right)+\gamma\right)G_0\left(u\right)=-\mu\pi_{1,1} \quad (6.1)$$

Then (6.1) can be written as

$$G_0^1\left(u\right)-\left(\frac{\lambda b_1}{\xi}+\frac{\gamma}{\xi\left(1-u\right)}\right)G_0\left(u\right)=-\frac{\mu\pi_{1,1}}{\xi\left(1-u\right)} \quad (6.2)$$

Multiplying this equation by $e^{-\frac{\lambda b_1}{\xi}u}\left(1-u\right)^{\frac{\gamma}{\xi}}$ and integrating from 0 to u, we get

$$e^{-\frac{\lambda b_1}{\xi}u}\left(1-u\right)^{\frac{\gamma}{\xi}}G_0\left(u\right)=-\frac{\mu\pi_{1,1}}{\xi}\int_0^u e^{-\frac{\lambda b_1}{\xi}y}\left(1-y\right)^{\frac{\gamma}{\xi}-1}dy+c \quad (6.3)$$

Put u =0 in this equation to get $G_0\left(0\right)=C$

$$G_0\left(u\right)=e^{\frac{\lambda b_1}{\xi}z}\left(1-u\right)^{\frac{\gamma}{\xi}}G_0\left(0\right)-\frac{\mu\pi_{1,1}}{\xi}e^{\frac{\lambda b_1}{\xi}u}\left(1-u\right)^{-\frac{\gamma}{\xi}}\int_0^u e^{-\frac{\lambda b_1}{\xi}y}\left(1-y\right)^{\frac{\gamma}{\xi}-1}dy \quad (6.4)$$

Since $G_0\left(1\right)>0$ and $\underset{u\to 1}{Lt}\left(1-u\right)^{\frac{\gamma}{\xi}}=0$, we must have

$$G_0\left(0\right)=\frac{\mu P_{1,1}}{\xi}\int_0^1 e^{-\frac{\lambda b_1}{\xi}y}\left(1-y\right)^{\frac{\gamma}{\xi}-1}dy \tag{6.5}$$

Let $K=\int_0^1 e^{-\frac{\lambda b_1}{\xi}y}\left(1-y\right)^{\frac{\gamma}{\xi}-1}dy$. (6.6)

Then $G_0\left(0\right)=\pi_{0,0}=\frac{\mu\pi_{1,1}}{\xi}K$. (6.7)

$$G_0\left(u\right)=e^{\frac{\lambda b_1}{\xi}u}\left(1-u\right)^{-\frac{\gamma}{\xi}}G_0\left(0\right)\left[1-\frac{\int_0^u e^{-\frac{\lambda b_1}{\xi}y}\left(1-y\right)^{\frac{\gamma}{\xi}-1}dy}{\int_0^1 e^{-\frac{\lambda b_1}{\xi}y}\left(1-y\right)^{\frac{\gamma}{\xi}-1}dy}\right]$$

From equation (6.5)

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The probability that the sever is on vacation period is given by $\sum_{n=0}^{\infty}\pi_{0,1}=G_0\left(1\right)$

$$G_0\left(1\right)=e^{\frac{\lambda b_1}{\xi}}G_0\left(0\right)\underset{u\to 1}{Lt}\frac{\left[1-\frac{\int_0^u e^{-\frac{\lambda b_1}{\xi}y}\left(1-y\right)^{\frac{\gamma}{\xi}-1}dy}{\int_0^1 e^{-\frac{\lambda b_1}{\xi}y}\left(1-y\right)^{\frac{\gamma}{\xi}-1}dy}\right]}{\left(1-u\right)^{\frac{\gamma}{\xi}}}$$

$$G_0\left(1\right)=\frac{\xi}{\gamma K}\pi_{0,0}$$

6.8)

Use (6.8) in (6.9) to get

$$G_0\left(1\right)==\frac{\mu\pi_{1,1}}{\gamma}$$

(6.9)

We multiply equation (5.3) by u and equation (5.4) by u^m and summing over m from 1 to ∞, we have

$$\left[\left(\lambda b_2 u-\mu\right)\left(1-u\right)+\alpha u\right]G_1\left(u\right)=\gamma u\,G_0\left(u\right)+\eta\,u\,G_2\left(u\right)-\left[\frac{\mu\gamma}{\lambda b_2}\left(1-u\right)-\frac{\alpha\gamma u}{\lambda b_2}+\frac{\xi}{K}u\right]\pi_{0,0} \tag{6.10}$$

and evaluating at u=1 in (6.11)

$$G_1(1)=\frac{\eta\, G_2(1)}{\alpha}+\frac{\alpha\gamma}{\lambda b_2}\pi_{0,0} \qquad (6.11)$$

We multiply equation (5.5) by u and equation (5.6) by u^m $(m\geq 2)$, summing over m from 1 to ∞

$$\xi\, u(1-u)G_2^{|}(u)-\left[\lambda b_3 u(1-u)+\xi(1-u)+\eta\, u\right]G_2(u)+\alpha u\,\pi_{1,0}=-\alpha u\, G_1(u) \qquad (6.12)$$

Substitute $G_1(u)$ from (6.11) in (6.12), to get

$$G_2^{|}(u)-\left(\frac{\lambda b_3}{\xi}+\frac{1}{u}+\frac{\eta}{\xi(1-u)}-\frac{\alpha\,\eta\, u}{\xi(1-u)\left((\lambda b_2-\mu)(1-u)+\alpha u\right)}\right)G_2(u)=$$

$$\frac{\alpha\pi_{1,0}\left[(\lambda b_2 u-\mu)(1-u)+\alpha u\right]-\alpha\left[\gamma u G_0(u)-\left\{\frac{\mu\gamma}{\lambda b_3}(1-u)-\frac{\alpha\gamma}{\lambda b_3}u+\frac{\xi}{K}u\right\}\pi_{0,0}\right]}{\xi(1-u)\left(\lambda b_2\, u(1-u)-\mu(1-u)+\alpha u\right)}, \qquad (6.13)$$

Multiplying equation (6.13) by

$$e^{-\frac{\lambda b_3}{\xi}u}\,\frac{1}{u}\left(-\lambda b_2 u^2+(\lambda b_2+\mu+\alpha)u-\mu\right)^{\frac{\eta}{2\xi}}\left(\frac{p+u-\theta}{p-u+\theta}\right)^{\frac{\eta(\alpha+\lambda b_2-\mu)}{4\xi p}}$$ on both sides and integrating,

where $p=\sqrt{\left(\frac{\lambda b_2+\mu+\alpha}{2\lambda b_2}\right)^2-\frac{\mu}{\lambda b_2}}\quad \theta=\frac{\lambda b_2+\mu+\alpha}{2\lambda b_2}$

we get

$$G_2(u)=\frac{\alpha}{\xi}e^{\frac{\lambda b_3}{\xi}u}u\left(-\lambda b_2 u^2+(\lambda b_2+\mu+\alpha)u-\mu\right)^{-\frac{\eta}{2\xi}}\left(\frac{p+u-\theta}{p-u+\theta}\right)^{\eta\frac{(\alpha+\lambda b_2-\mu)}{4\xi p}}H(u)\pi_{0,0} \qquad (6.14)$$

Where

$$H(u)=\int_0^u e^{-\frac{\lambda b_3 y}{\xi}}\left(-\lambda b_2 y^2+(\lambda b_2+\mu+\alpha)y-\mu\right)^{\frac{\eta}{2\xi}}\left(\frac{p+u-\theta}{p-u+\theta}\right)^{\eta\frac{(\alpha+\lambda b_2-\mu)}{4\xi p}}R(y)\,dy.$$

With

$$R(y)=\frac{\gamma y(1-y)-\gamma y e^{\frac{\lambda b_1}{\xi}y}F(x)+\frac{\xi}{K}y}{(1-y)\left(\lambda b_2 y(1-y)-\mu(1-y)+ay\right)}.$$

$$F(u)=(1-u)^{-\frac{\gamma}{\xi}}\left[1-\frac{\int_0^u e^{-\frac{\lambda b_3}{\xi}y}(1-y)^{\frac{\gamma}{\xi}-1}dy}{\int_0^1 e^{-\frac{\lambda b_3}{\xi}y}(1-y)^{\frac{\gamma}{\xi}-1}dy}\right]=(1-u)^{-\frac{\gamma}{\xi}}\left[1-\frac{\int_0^u e^{-\frac{\lambda b_3}{\xi}y}(1-y)^{\frac{\gamma}{\xi}-1}dy}{k}\right].$$

Put u=1 in (3.17)

$$G_2(1)=\frac{\alpha}{\xi}e^{\frac{\lambda b_3}{\xi}}(\alpha)^{-\frac{\eta}{2\xi}}\left(\frac{p-\theta+1}{p+\theta-1}\right)^{\eta\frac{(\alpha+\lambda b_2-\mu)}{4\xi p}}H(1)\pi_{0,0}. \qquad (6.15)$$

Using the normalizing condition

$G_0(1)+G_1(1)+G_2(1)=1,$

We obtain

$$\pi_{0,0}=\left[\frac{\xi}{\gamma K}+\frac{\gamma}{\lambda b_1}+\left(1+\frac{\eta}{\alpha}\right)\frac{\alpha}{\xi}e^{\frac{\lambda b_3}{\xi}}(\alpha)^{-\frac{\eta}{2\xi}}\left(\frac{p-\theta+1}{p+\theta-1}\right)^{\eta\frac{(\alpha+\lambda b_3-\mu)}{4\xi p}}H(1)\right]^{-1} \quad (6.16)$$

We differentiate (6.2) w.r.to' u' and evaluating at u=1, then

$$G_0^{|}(1)=\frac{\lambda b_1}{\gamma+\xi}G_0(1). \quad (6.17)$$

Differentiating (6.11) w. r. t. 'u' and evaluating at u=1, we have

$$G_1^{|}(1)=\frac{(\lambda b_2-\mu)}{\alpha}G_1(1)+\frac{\gamma}{\alpha}G_0^{|}(1)+\frac{\eta}{\alpha}G_2^{|}(1)+\frac{\eta}{\alpha}G_2(1)+\frac{(\mu+\alpha)\gamma}{\lambda b_2}\pi_{0,0} \quad (6.18)$$

Differentiating (6.13) w. r. t. 'u' and evaluating at u=1, we have

$$G_2^{|}(1)=\frac{\left[\eta(\lambda b_2-\mu+\alpha)+\alpha(\lambda b_3+\xi)\right]G_2(1)+\alpha\gamma G_0^{|}(1)+\alpha\gamma\left(1-\frac{\alpha}{\lambda b_2}\right)\pi_{0,0}}{\alpha\xi}. \quad (6.19)$$

Performance measures

(i) In vacation period of server, the average number of jobs waiting in the queue is

$$E(L_0)=G_0^{|}(1)=\frac{\lambda b_1\xi}{(\xi+\gamma)\gamma K}\pi_{0,0}. \quad (6.20)$$

(ii) In busy period of server, the average number of jobs waiting in the queue is

$$E(L_1)=G_1^{|}(1)=\frac{(\lambda b_2-\mu)}{\alpha}G_1(1)+\frac{\gamma}{\alpha}G_0^{|}(1)+\frac{\eta}{\alpha}G_2^{|}(1)+\frac{\eta}{\alpha}G_2(1)+\frac{(\mu+\alpha)\gamma}{\lambda b_2}\pi_{0,0} \quad (6.21)$$

(iii) In breakdown period and repair period of system, average number of jobs waiting in the queue is

$$E(L_2)=G_2^{|}(1)=\frac{\left[\eta(\lambda b_2-\mu+\alpha)+\alpha(\lambda b_3+\xi)\right]G_2(1)+\alpha\gamma G_0^{|}(1)+\alpha\gamma\left(1-\frac{\alpha}{\lambda b_2}\right)\pi_{0,0}}{\alpha\xi}. \quad (6.22)$$

(iv) The mean system size is

$$E(L)=E(L_0)+E(L_1)+E(L_2) \quad (6.23)$$

(v) By Little's formula mean waiting time in the system is

$$E(W)=\frac{E(L)}{\lambda} \quad (6.24)$$

(vi) The proportion of jobs served per unit time is

$$P_s=\frac{\mu G_1(1)}{\lambda}=\frac{\theta}{(\xi+\theta)(1-r)}G_0(1) \quad (6.25)$$

(vii) The proportion of jobs reneged per unit time is

$R_m = \sum_1^\infty m\,\xi\,\pi_{0,m}$. (6.26)

(viii) The Proportion of jobs balked per unit time is

$$R_{balk} = \sum_{m=0}^{\infty} \lambda\left(1-b_0\right)\pi_{0,m} + \sum_{m=0}^{\infty} \lambda\left(1-b_1\right)\pi_{1,m} + \sum_{m=0}^{\infty} \lambda\left(1-b_2\right)\pi_{2,m} + \sum_{m=0}^{\infty} \lambda\left(1-b_3\right)\pi_{3,m} \quad (6.27)$$

7. CONCLUSIONS

The present paper investigated the M/M/1 queuing system with customer reneging and balking. Other conditions of this system are, an unreliable server with vacation under both the single and multiple vacation cases. We have obtained the expressions for the probability generating functions for the number of customers in the system for the different states of the server including vacation or busy or breakdown periods. We also obtained the closed form expressions for various performance measures.

REFERENCES

1. Dequan Yue, Wuyi Yue & Xiuju Li(2011), "Analysis of a two-phase queuing system with impatient customers and multiple vacations", In: The Tenth International Symposium on Operations Research and Its Applications, 2011, 292-298.
2. Sherif I. Ammar, "Transient analysis of an M/M/1 queue with impatient behavior and
3. multiple vacations" Applied mathematics and Computation, Vol.260, 2015, 97-105.
4. S. Hanumantha Rao, V. Vasanta Kumar, B. Srinivasa Kumar, T. Srinivasa Rao, "Analysis of Two-Phase Queuing System with Impatient Customers, Server Breakdowns and Delayed Repair, International Journal of Pure and Applied Mathematics Volume 115 No. 4 2017, 651-663.
5. E. Altman, U. Yechaili, Analysis of customers impatience with server vacations", Queueing Systems, 52 (2006), 261-279.
6. Wang, K., Li, N., Jiang, Z."Queueing System with Impatient Customers: A Review" In: Proceddings of 2010 IEEE International Conference on Servuce Operations and Logistics and Informatics, pp.82-87(2010) .