Bondage Number of Lexicographic Product of Two Graphs

Deepak. G., Indiramma. M. H., Bindu. M.G.

***Abstract*: The bondage number b(G) of a nonempty graph G is the cardinality of a smallest set of edges whose removal from G results in a graph with a domination number greater than the domination number of G. In this paper, we study the bondage number of the Lexicographic product of two paths, Lexicographic product of path and a graph with given maximum degree.**

***Index Terms*: Graph, Lexicographic product, Domination number, Bondage number. 2010 Mathematics Subject Classification. 05C38, 05C69, 05C76.**

I. INTRODUCTION

Unless mentioned otherwise for terminology and notation the reader may refer F. Harary [7], new ones will be introduced as and when found necessary. Let $G = (V(G), E(G))$ be a finite, simple and connected graph, where $V(G)$ is the vertex set and $E(G)$ is the edge set. The neighborhood of a vertex $v \in V(G)$, denoted by $N_G(V)$, is the set of vertices adjacent to v in G. Denote $E_G(v)$ to be the set of edges incident with v in G. The closed neighborhood of a vertex v in a graph G is $N_G[v] = N_G(v) \cup \{v\}$. The degree $d_G(v)$ of a vertex v in G is the number of edges of G incident with v. Denote $\delta(G)$ and $\Delta(G)$ to be the minimum and maximum degrees, respectively, of vertices of G. A vertex of degree zero is called an isolated vertex. An edge incident with a vertex of degree one is called a pendant edge. A subset $S \subseteq V(G)$ of vertices is a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex of S. The domination number $\gamma(G)$ is the minimum cardinality of all dominating sets in G. The study of domination and related properties is one of the fastest growing areas in graph theory and also is frequently used to study property of networks. For a detailed study of domination one can see [11], [12] and [13]. In 1990, Fink et al. [4] have introduced the concept of bondage number of a graph. The bondage number $b(G)$ of a nonempty graph G is the cardinality of a smallest set of edges whose removal from G results in a graph with domination number greater than $\gamma(G)$. In 1990, Fink et al. [4] have obtained the bondage number of cycles, paths and complete multipartite graphs and have obtained a bound $b(T) \leq 2$ for any tree T. In [8], Hartnell and Rall have characterized trees with bondage number 2. In [9], Hartnell and Rall have proved $b(Gn) = 3/4\,\Delta$, for the cartesian product $Gn = Kn \square Kn$, $n > 1$. In [14], Hu and Xu have determined the bondage numbers of Cartesian product of two paths Pn and Pm for $n \geq 2$, $m \leq 4$.

In [16], Kang et al. have proved $b(C_n \square C_4) = 4$, $n \geq 4$ for discrete torus $C_n \square C_4$.

Definition 1.1. [6] Given graphs G and H, the lexicographic product $G[H]$ has vertex set $\{(g, h) : g \in V(G), h \in V(H)\}$ and two vertices $(g, h), (g', h')$ are adjacent if and only if either $[g, g']$ is an edge of G or $g = g'$ and $[h, h']$ is an edge of H.

Theorem 1.1. [1] If G is a graph of order $m \geq 2$ with $\Delta(G) = m - 1$ then
$\gamma(Pn[G]) = \lceil n/3 \rceil$, $n \geq 2$.

II. BONDAGE NUMBER OF LEXICOGRAPHIC PRODUCT OF TWO GRAPHS

Theorem 2.1. If a graph G of order m has at most one vertex of degree m-1 then $b(P_n[G]) = 1$, where $n = 3k$, $k \geq 1$.

Proof. Let a graph G of order m, labeled as $v_1, v_2, \ldots, v_k, \ldots, v_m$ has at most one vertex v_k of degree $m - 1$ and P_n be a path on n vertices, labeled as $u_1, u_2, \ldots, u_n$.

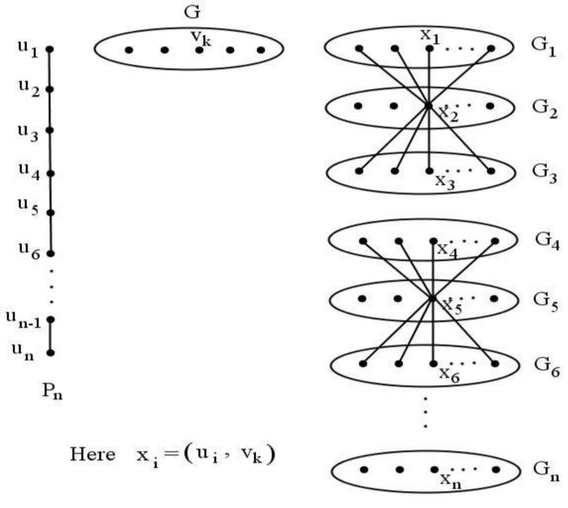

Figure 1. Bondage Number of $P_n[G]$

Let G_i, $1 \leq i \leq n$ be n copies of the graph G, substituted in the places of the vertices u_i, $1 \leq i \leq n$, respectively, in the lexicographic product $P_n[G]$, as shown in Figure 1.

In $P_n[G]$, denote $x_1 = (u_1, v_k) \in G_1$, $x_2 = (u_2, v_k) \in G_2$, $x_3 = (u_3, v_k) \in G_3, \ldots, x_n = (u_n, v_k) \in G_n$, as shown in Figure 1. The set $S = \{x_{3t-1} / 1 \leq t \leq k\}$ form a unique minimum dominating set in $P_n[G]$ and also $N[x_i] \cap N[x_j] = \Phi$ for every $x_i, x_j \in S$. Therefore, the removal of any edge (say, e) incident with any of the vertex in S increases the domination number, i.e.,

$\gamma(P_n[G] - e) > \gamma(P_n[G])$.

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Dr. Deepak. G, Department of Mathematics, Sri Venkateswara College of Engineering , Bangalore, India.

Indiramma. M. H., Department of Mathematics, Sri Venkateswara College of Engineering , Bangalore, India.

Bindu. G, Department of Mathematics, Sapthagiri College of Engineering, Bangalore, India.

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Hence, $b(Pn[G]) = 1$.

Theorem 2.2. If a graph G of order m has at most one vertex of degree $m - 1$ then $b(P_n[G]) = 3$, where $n = 3k + 1, k \geq 2$.

Proof. Let a graph G of order m, labeled as $v_1, v_2, \ldots, v_k, \ldots, v_m$, has at most one vertex v_k of degree $m - 1$ and P_n be a path on $n = 3k + 1, k \geq 2$ vertices, labeled as $u_1, u_2, \ldots, u_n$.

Let $G_i, 1 \leq i \leq n$ be n copies of the graph G substituted in the places of the vertices $u_i, 1 \leq i \leq n$, respectively, in $P_n[G]$ and denote $x_1 = (u_1, v_k) \in G_1$, $x_2 = (u_2, v_k) \in G_2$, $x_3 = (u_3, v_k) \in G_3, \ldots, x_n = (u_n, v_k) \in G_n$, as shown in Figure 2.

Let x, y, z be any three vertices in G_{n-5}, G_{n-3} and G_{n-1}, respectively. From Theorem [1.1], the domination number $\gamma(P_n[G]) = k + 1$.

List of all possibilities of a minimum dominating set containing the vertices from G_{n-6}, G_{n-5}, G_{n-4}, G_{n-3}, G_{n-2}, G_{n-1} and G_n is as follows.

$D_1 : x_{n-4}, x_{n-1}$,
$D_2 : x_{n-5}, x_{n-3}, x_{n-1}$,
$D_3 : x_{n-5}, x_{n-4}, x_{n-1}$,
$D_4 : x_{n-5}, x_{n-2}, x_{n-1}$,
$D_5 : x_{n-5}, x_{n-2}, x_n$.

We now prove that, the removal of three edges xv_{n-5}, yv_{n-3}, zv_{n-1} increases the domination number in all possible vertex distributions of minimum dominating set. Here five cases arise.

Case (1): $D_1 : x_{n-4}, x_{n-1}$. That is, the minimum dominating set contains the vertices $x_{n-4} \in G_{n-4}$ and $x_{n-1} \in G_{n-1}$ and contains no vertex from G_{n-6}, G_{n-5}, G_{n-3}, G_{n-2}, G_n.

In this case, the removal of three edges xx_{n-5}, yx_{n-3}, zx_{n-1} leaves the vertex z undominated by any of the vertices of a minimum dominating set with this possibility of distribution of vertices. Hence, the domination number will be increased in this case, i.e.,

$\gamma (P_n[G] - \{xx_{n-5}, yx_{n-3}, zx_{n-1}\}) > \gamma(P_n[G])$.

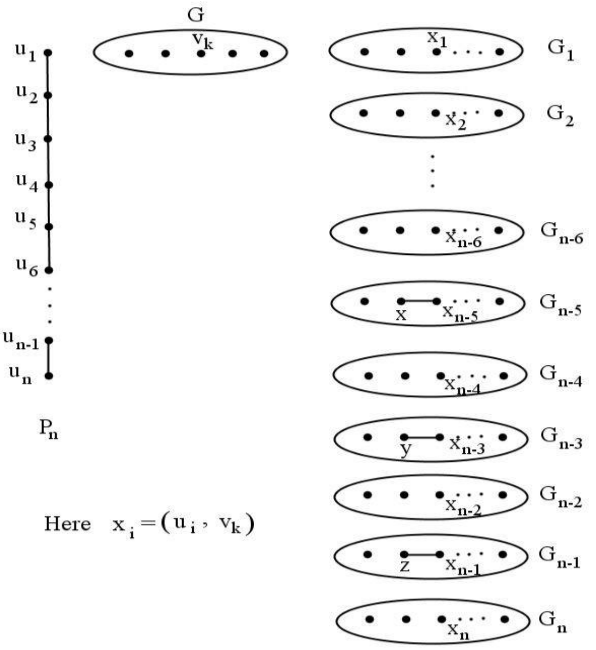

Figure 2. Bondage Number of $P_n[G]$

Case (2): $D_2 : x_{n-5}, x_{n-3}, x_{n-1}$. That is, the minimum dominating set contains the vertices $x_{n-5} \in G_{n-5}$, $x_{n-3} \in G_{n-3}$ and $x_{n-1} \in G_{n-1}$ and contains no vertex from G_{n-6}, G_{n-4}, G_{n-2}, G_n. In this case, the removal of three edges xx_{n-5}, yx_{n-3}, zx_{n-1} leaves the vertices x, y, and z undominated by any of the vertices of a minimum dominating set with this possibility of distribution of vertices. Hence, the domination number will be increased in this case, i.e.,

$\gamma(P_n[G] - \{xx_{n-5}, yx_{n-3}, zx_{n-1}\}) > \gamma(P_n[G])$.

Case (3): $D_3 : x_{n-5}, x_{n-4}, x_{n-1}$. That is, the minimum dominating set contains the vertices $x_{n-5} \in G_{n-5}$, $x_{n-4} \in G_{n-4}$ and $x_{n-1} \in G_{n-1}$ and contains no vertex from G_{n-6}, G_{n-5}, G_{n-3}, G_{n-2}, G_n.

In this case, the removal of three edges xx_{n-5}, yx_{n-3}, zx_{n-1} leaves the vertex z undominated by any of the vertices of a minimum dominating set with this possibility of distribution of vertices. Hence, the domination number will be increased in this case, i.e.,

$\gamma(P_n[G] - \{xx_{n-5}, yx_{n-3}, zx_{n-1}\}) > \gamma(P_n[G])$.

Case (4): $D_4 : x_{n-5}, x_{n-2}, x_{n-1}$. That is, the minimum dominating set contains the vertices $x_{n-5} \in G_{n-5}$, $x_{n-2} \in G_{n-2}$ and $x_{n-1} \in G_{n-1}$ and contains no vertex from G_{n-6}, G_{n-4}, G_{n-3}, G_n.

In this case, the removal of three edges xx_{n-5}, yx_{n-3}, zx_{n-1} leaves the vertex x undominated by any of the vertices of a minimum dominating set with this possibility of distribution of vertices. Hence, the domination number will be increased in this case, i.e.,

$\gamma(P_n[G] - \{xx_{n-5}, yx_{n-3}, zx_{n-1}\}) > \gamma(P_n[G])$.

Case (5): $D_5 : x_{n-5}, x_{n-2}, x_n$. That is, the minimum dominating set contains the vertices $x_{n-5} \in G_{n-5}$, $x_{n-2} \in G_{n-2}$ and $x_n \in G_n$ and contains no vertex from G_{n-6}, G_{n-4}, G_{n-3}, G_{n-1}.

In this case, the removal of three edges xx_{n-5}, yx_{n-3}, zx_{n-1} leaves the vertex x undominated by any of the vertices of this distribution. Hence, the domination number will be increased in this case also, i.e.,

$\gamma(P_n[G] - \{xx_{n-5}, yx_{n-3}, zx_{n-1}\}) > \gamma(P_n[G])$. Hence, $b(P_n[G]) = 3$, $n = 3k + 1, k \geq 2$.

Theorem 2.3. If a graph G of order $m \geq 3$ has at most one vertex of degree $m - 1$ then $b(P_n[G]) = 1$, where n = 3k + 2, k≥ 1.

Proof. Let a graph G of order $m \geq 3$, labelled as $v_1, v_2, \ldots, v_k, \ldots, v_m$, has at most one vertex of degree $m - 1$ and P_n be a path on $n = 3k + 2, k \geq 1$ vertices, labeled as $u_1, u_2, \ldots, u_n$. The lexicographic product $P_n[G]$ is as shown in Figure 3.

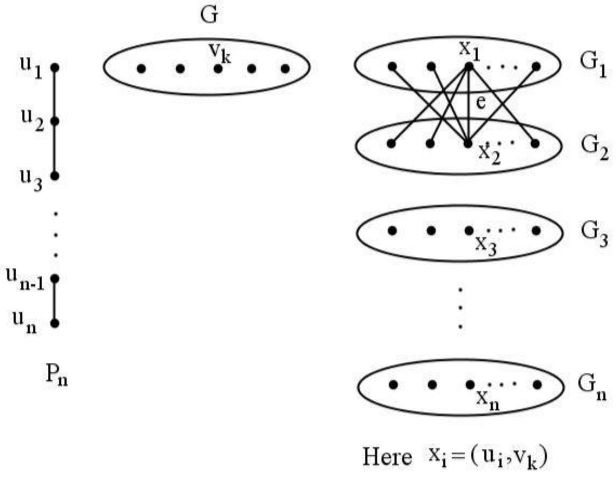

Figure 3. Bondage Number of $P_n[G]$

Clearly, every minimum dominating set contains a vertex $x_1 = (u_1, v_k)$ or $x_2 = (u_2, v_k)$. By removing an edge e between the vertices x_1 and x_2, as shown in the Figure 3, the vertex x_1 or x_2 remains undominated by every minimum dominating set. Therefore, $\gamma(P_n[G] - e) > \gamma(P_n[G])$. Hence, $b(P_n[G]) = 1$.

Theorem 2.4. For any path P_n, $n \geq 2$

$$b(P_2[P_n]) = \begin{cases} 2, & if \quad n = 2 \\ 1, & if \quad n = 3 \\ 6, & if \quad n = 4 \\ n+1, & if \quad n \geq 5 \end{cases}$$

Proof. Let $P_2 : u_1, u_2$ be a path on two vertices and $P_n : v_1, v_2, \ldots, v_n$ be a path on $n \geq 2$ vertices. Here four cases arise.

Case (1): $n = 2$.

The lexicographic product $P_2[P_2]$ is as shown in Figure 4.

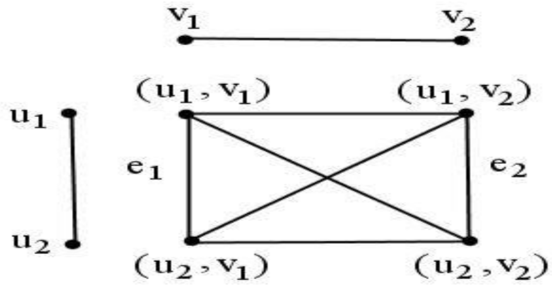

Figure 4. Bondage Number of $P_2[P_2]$

Since $P_2[P_2]$ is a complete graph on four vertices, removal of one edge does not increase the domination number. The removal of two edges $e_1 = [(u_1, v_1), (u_2, v_1)]$ and $e_2 = [(u_1, v_2), (u_2, v_2)]$ increases the domination number, i.e., $\gamma(P_2[P_n] - \{e_1, e_2\}) > \gamma(P_2[P_n])$. Hence, $b(P_2[P_n]) = 2$, for $n = 2$.

Case (2): $n = 3$.

The lexicographic product $P_2[P_3]$ is as shown in Figure 5.

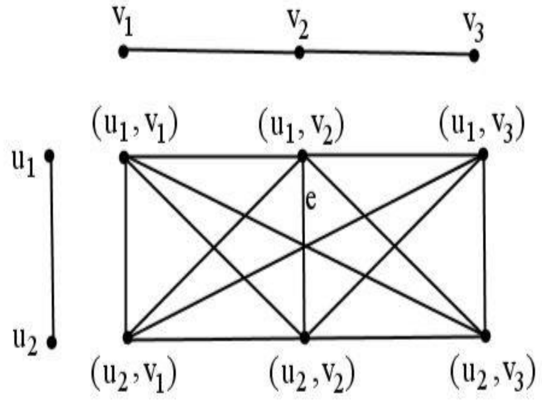

Figure 5. Bondage Number of $P_2[P_3]$

The sets $\{(u_1, v_2)\}$ and $\{(u_2, v_2)\}$ are the only two minimum dominating sets in the lexicographic product $P_2[P_3]$. Therefore, the removal of an edge 'e' between the vertices (u_1, v_2) and (u_2, v_2), increases the domination number, i.e., $\gamma(P_2[P_n] - \{e\}) > \gamma(P_2[P_n])$, $n = 3$. Hence, $b(P_2[P_n]) = 1$, for $n = 3$.

Case (3): $n = 4$.

The lexicographic product $P_2[P_4]$ is as shown in Figure 6.

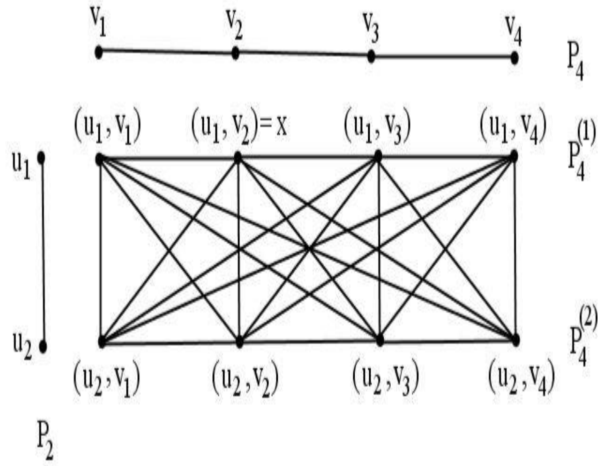

Figure 6. Bondage Number of $P_2[P_4]$

First we prove, the removal of atmost five edges from $P_2[P_4]$ does not increase the domination number.

Let F be a set of atmost five edges in $P_2[P_4]$. Here three subcases arise.

Subcase (3.1): If F contains at most three edges, then there exist a vertex x in P_4^1 and y in P_4^2, which dominates all the vertices of $P_2[P_4] - F$. Hence, $\{x, y\}$ is the minimum dominating set.

Subcase (3.2): Suppose F contains four edges.

Subcase (3.21): F contains at least one edge from P_4^1 or P_4^2. There exists a vertex, say x, in P_4^1 which dominates all the vertices of P_4^2 and there exists a vertex, say y, in P_4^2 which dominates all the vertices of P_4^1. Hence, $\{x, y\}$ is the minimum dominating set of $P_2[P_n] - F$.

Subcase (3.22): Suppose, F contains four independent edges from $E(P_4^1 : P_4^2)$,then the pair of vertices incident with any of the edge in *F* dominates all the vertices of $P_2[P_4] - F$.

Now suppose, *F* contains four nonindependent edges (i.e., at least two edges of *F* have same end vertex) from $E(P_4^1 : P_4^2)$, then $P_2[P_4] - F$ contains a vertex, say, *x* in P_4^1 or P_4^2, which is adjacent to all the vertices of P_4^2 or P_4^1, as the case may be. For suppose $P_2[P_4] - F$ contains a vertex $x \in P_4^1$, adjacent to all the vertices of P_4^2. By taking a vertex $y \in G_1$ which is not adjacent to the vertex *x*, we get a minimum dominating set $\{x, y\}$ in $P_2[P_4] - F$.

Subcase (3.3): Suppose *F* contains five edges.

Subcase (3.31): If *F* contains one edge from $P_4^{(1)}$ or $P_4^{(2)}$ then the remaining four edges will be from $E(P_4^{(1)} : P_4^{(2)})$. From Subcase(3.2), the domination number is 2.

If *F* contains at least two edges from $P_4^{(1)}$ or $P_4^{(2)}$ then there exists a vertex, say *x*, in $P_4^{(1)}$, which dominates all vertices of $P_4^{(2)}$ and there exists a vertex, say *y*, in $P_4^{(2)}$, which dominates all the vertices of $P_4^{(1)}$. Hence, $\{x, y\}$ is the minimum dominating set of $P_2[P_n] - F$.

Subcase (3.32): *F* contains five edges from $E(P_4^{(1)} : P_4^{(2)})$. Suppose $P_4^{(1)}$ or $P_4^{(2)}$ contains a vertex, say, *x*, to which no edge of *F* is incident in $P_2[P_4]$. For suppose $x \in P_4^{(1)}$ is the vertex to which no edge of *F* is incident. Along with a vertex *x*, by taking a vertex $y \in P_4^{(1)}$ not adjacent with a vertex *x*, we get a minimum dominating set $\{x, y\}$ in $P_2[P_4] - F$.

Now, suppose every vertex of $P_4^{(1)}$ or $P_4^{(2)}$ is incident with at least one edge of *F*. There exists a vertex $x_1^j = (u_1, v_j)$ in $P_4^{(1)}$ to which two edges of *F* are incident. The vertex set $\{(u_1, v_k), (u_2, v_k)\}$ where (u_1, v_k) is adjacent to (u_1, v_j), is the minimum dominating set in $P_2[P_4] - F$. Hence, we have proved that $b(P_2[P_4]) \not\leq 5$. Therefore, $b(P_2[P_4]) > 5$.

Removal of all six edges incident with a vertex (u_1, v_2), increases the domination number. Hence, $b(P_2[P_n]) = 6$, $n = 4$.

Case(4): $n \geq 5$.

The lexicographic product $P_2[P_n]$, $n \geq 5$ is as shown in Figure 7.

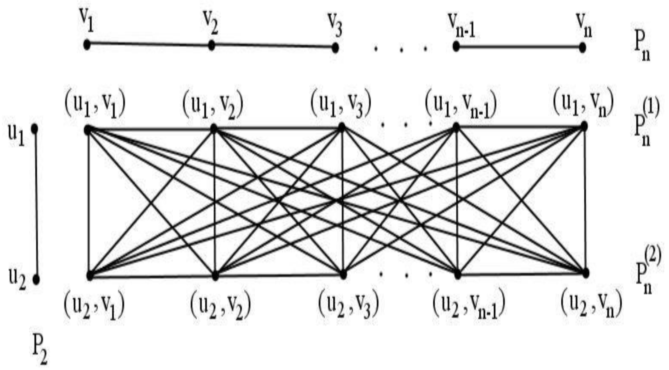

Figure 7. Bondage Number of $P_2[P_n]$

First, we prove that, the removal of at most *n* edges does not increase the domination number.

Let *F* be a set of at most *n* edges in $P_2[P_n]$.

Subcase (4.1): *F* contains at least one edge from $P_n^{(1)}$ or $P_n^{(2)}$.

In this case, there exist a vertex, say *x*, in $P_n^{(1)}$, which dominates all vertices of $P_n^{(2)}$ and there exist a vertex, say y, in $P_n^{(2)}$, which dominates all vertices of $P_n^{(1)}$. Hence, $\{x, y\}$ is the minimum dominating set of $P_2[P_n] - F$.

Subcase (4.2): *F* contains edges from $E(P_n^{(1)} : P_n^{(2)})$ only.

Subcase (4.21): If all the *n* edges of *F* are independent then the pair of end vertices of an edge $e \in F$, form a minimum dominating set of $P_2[P_n] - F$.

Subcase (4.22): Suppose the edges of *F* are not independent. If every vertex of $P_n^{(1)}$ is incident with an edge of *F* then there exists a vertex $y \in P_n^{(2)}$, such that no edge of *F* is incident with *y* and the vertex *y* has a adjacent vertex $x \in P_n^{(2)}$ to which at least one edge, say *e*, of *F* is incident. The set $\{x, z\}$ where $z \in P_n^{(1)}$ is a vertex incident with *e*, form a minimum dominating set in $P_2[P_n] - F$.

If every vertex of $P_n^{(2)}$ is incident with an edge of *F*, then there exists a vertex $v \in P_n^{(1)}$ such that no edge of *F* is incident with *v* and the vertex *v* has an adjacent vertex $u \in P_n^{(1)}$ to which at least one edge, say, *e* of *F* is incident. The set $\{u, w\}$, where $w \in P_n^{(2)}$ is a vertex incident with an edge *e*, is a minimum dominating set.

If there is a vertex $x \in P_n^{(1)}$, to which no edge of *F* is incident and there is a vertex $y \in P_n^{(2)}$ to which no edge of *F* is incident then the set $\{x, y\}$ is the minimum dominating set. Hence, $b(P_2[P_n]) > n$.

By removing the $n + 1$ edges incident to (u_1, v_1) the domination number of $P_2[P_n] - F$ will be increased. Therefore, $b(P_2[P_n]) = n + 1$, for $n \geq 5$.

Theorem 2.5. For a path P_n, $n \geq 2$,

$$b(P_3[P_n]) = \begin{cases} 1, & \text{if } n = 2,3 \\ n+1, & \text{if } n \geq 4 \end{cases}$$

Proof. Let $P_n : v_1, v_2, \ldots, v_n$ be a path on $n \geq 2$ vertices. Here three cases arise.

Case (1): $n = 2$.

In $P_3[P_2]$, as shown in Figure 8, the sets $\{(u_2, v_1)\}$ and $\{(u_2, v_2)\}$ are the only two minimum dominating sets. Removal of an edge e between the vertices (u_2, v_1) and (u_2, v_2), increases the domination number, i.e., $\gamma(P_3[P_2] - \{e\}) > \gamma(P_3[P_2])$. Hence, $b(P_3[P_2]) = 1$.

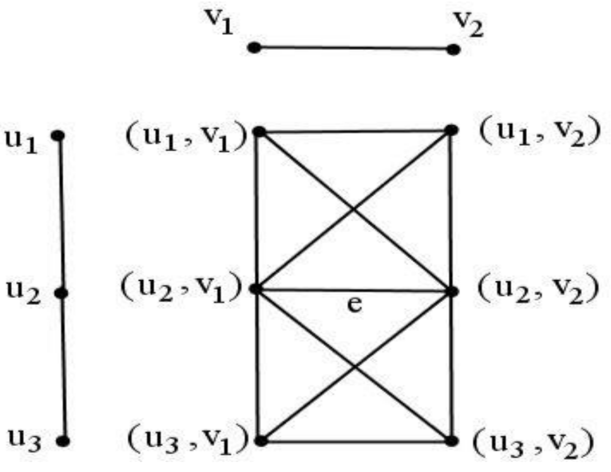

Figure 8. Bondage number of *P₃[P₂]*

Case (2): *n=3*

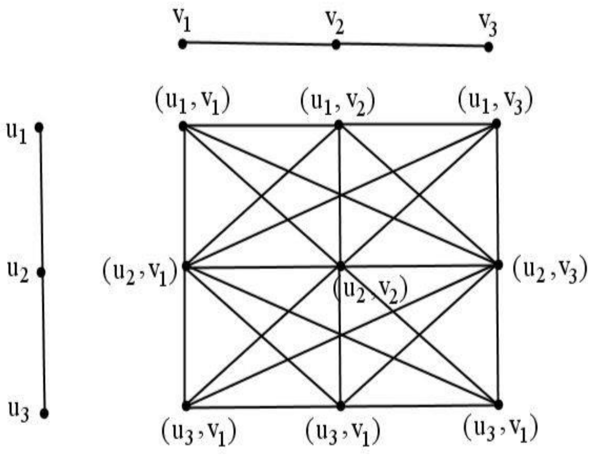

Figure 9. Bondage Number of P₃[P₃]

In $P_3[P_3]$, as shown in Figure 9, the set $\{(u_2, v_2)\}$ is the only minimum dominating set. Removal of any edge, say e, incident with a vertex (u_2, v_2) increases the domination number, i.e., $\gamma(P_3[P_3]-\{e\}) > \gamma(P_3[P_3])$. Hence, $b(P_3[P_3]) = 1$.

Case (3): $n \geq 4$.

The lexicographic product $P_3[P_n]$, $n \geq 4$ is as shown in Figure 10.

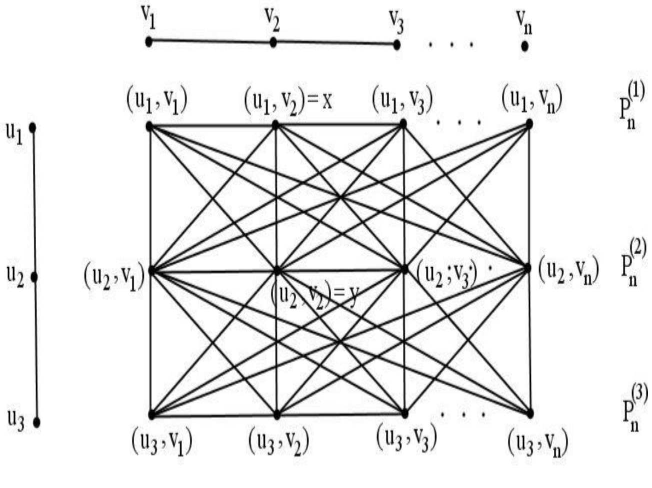

Figure 10. Bondage number of P₃[Pₙ]

The set $\{(u_1, v_1), (u_2, v_1)\}$ is the minimum dominating set. Hence, $\gamma(P_3[P_n]) = 2$.First we prove $b(P_3[P_n]) \nleq n$.

Let F be any set of at most n edges taken from $P_3[P_n]$.

Subcase (3.1): $|F| < n$.

There exist a vertex in $P_n^{(1)}$ which dominates all the vertices of $P_n^{(2)}$ and there exist a vertex in $P_n^{(2)}$ which dominates all the vertices of $P_n^{(1)}$ and $P_n^{(3)}$.Hence, $\gamma(P_3[P_n] - F) = 2$.

Subcase (3.2): $|F| = n$.

If there is a vertex, say $x_2^j \in P_n^{(2)}$ to which no edge of F is incident then x_2^j dominates all the vertices of $P_n^{(1)}$ and $P_n^{(3)}$. Since $|F| < 2n$, there exist a vertex y in $P_n^{(1)}$ or $P_n^{(3)}$ to which no edge of F is incident. Therefore, y dominates all the vertices of $P_n^{(2)}$. Hence, $\{x_2^j, y\}$ is the minimum dominating set. Hence, $\gamma(P_3[P_n] - F) = 2$.

Suppose every vertex of $P_n^{(2)}$ is incident with an edge of F. If all the edges of F are independent then there exist a vertex in $P_n^{(2)}$ which dominates all the vertices of $P_n^{(1)}$ or all the vertices of $P_n^{(3)}$. For suppose, there is a vertex, say x_2^k in $P_n^{(2)}$ which dominates all the vertices of $P_n^{(3)}$, then the set $\{x_1^j, x_2^k\}$, where $x_1^j \in P_n^{(1)}$ is a vertex such that the edge $x_1^j x_2^k$ belongs to F , dominates all the vertices of $P_3[P_n]$.

Now suppose the edges of F are not independent.

If there exist a vertex say $x_1^k \in P_n^{(1)}$ to which exactly one edge of F is incident then $\left\{x_1^k, x_2^l\right\}$, where $x_2^l \in P_n^{(2)}$ is a vertex such that the edge $x_1^k x_2^l \in F$, is the minimum dominating set.

Suppose $P_n^{(1)}$ contains no vertex, to which exactly one edge of F is incident. Let $x_1^k \in P_n^{(1)}$ be a vertex to which at least two edges of F are incident. Let $x_1^l \in P_n^{(1)}$ be a vertex adjacent to x_1^k such that no edge of F is incident with x_1^l and $x_2^r \in P_n^{(2)}$ is a vertex such that the edge $x_1^k x_2^r \in F$. The set $\left\{x_1^l, x_2^r\right\}$form a minimum dominating set. Hence, $\gamma(P_3[P_n] - F) = 2$. Hence, $b(P_3[P_n]) > n$.

The removal of $n + 1$ edges incident with $x_1^1 = (u_1, v_1)$ increases the domination number. Therefore, $b(P_3[P_n]) = n + 1$, $n \geq 4$

Theorem 2.6. If G is a connected graph of order $m \geq 4$ with $\Delta(G) \leq m-2$ then $b(P_n[G]) = m$, if $n = 4k$ or $4k + 1$, $k \geq 1$

Proof. Let G be a connected graph of order $m \geq 4$, labeled as $v_1, v_2, \ldots, v_n$ with $\Delta(G) \leq m-2$ and P_n be a path on n vertices, labeled as $u_1, u_2, \ldots, u_n$. Let $G_1, G_2, \ldots, G_n$ be n copies of the graph G substituted in the places of $u_1, u_2, \ldots, u_n$, respectively. Let $V(G_i) : x_i^1, x_i^2, \ldots\ldots\ldots, x_i^m$, as shown in Figure11.

Case (1): $n = 4k$.

Every minimum dominating set contains one vertex from each G_{4t-2}, $1 \leq t \leq k$ and one vertex from each G_{4t-1}, $1 \leq t \leq k$. There exists no minimum dominating set which contains two vertices from any G_i, $1 \leq i \leq n$.

First we prove, removal of the set F of at most $m - 1$ edges does not increase the domination number. In $P_n[G] - F$, there exist at least one vertex in each G_{4t-2}, $1 \leq t \leq k$ and one vertex from each G_{4t-1}, $1 \leq t \leq k$, to which no edge of F is incident, for every possible F . Hence, $\gamma(P_n[G] - F) = \gamma(P_n[G])$.

The removal of m independent edges from $E(G_1 : G_2)$ increases the domination number, as there exist no minimum dominating set which contains one vertex from G_1 and one vertex from G_2. Therefore, $\gamma(P_n[G] - F) > \gamma(P_n[G])$. Hence, $b(P_n[G]) = m$, where $n = 4k$.

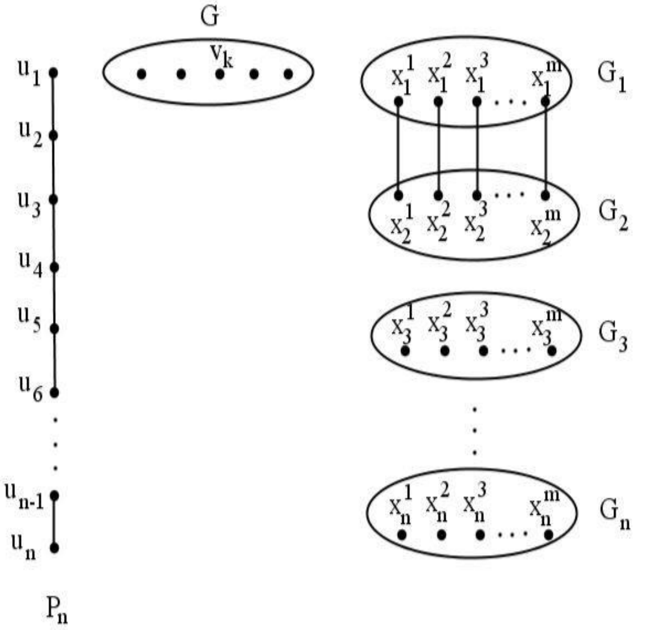

Figure 11. Bondage number of $P_n[G]$

Case (2): $n = 4k + 1$.

Every minimum dominating set contains one vertex from each G_{4t-2}, $1 \leq t \leq k$ and one vertex from each G_{4t-1}, $1 \leq t \leq k$ and one vertex from G_{n-1}. There exists no minimum dominating set which contains two vertices from any G_i, $1 \leq i \leq n$. As in Case(i), in $P_n[G] - F$, there exist at least one vertex in each G_{4t-2}, $1 \leq t \leq k$, one vertex from each G_{4t-1}, $1 \leq t \leq k$ and one vertex from each $Gn-1$, to which no edge of F is incident, for every possible F . Hence, $\gamma(P_n[G] - F) = \gamma(P_n[G])$. The removal of m independent edges from $E(G_1 : G_2)$ increases the domination number, as there exist no minimum dominating set which contains one vertex from G_1 and one vertex from G_2. Therefore, $\gamma(P_n[G] - F) > \gamma(P_n[G])$. Hence, $b(P_n[G]) = m$, where $n = 4k + 1$.

III. CONCLETION

In this paper we find the bondage number of the graphs $P_m[P_n]$, $n \geq 2$, $m = 2, 3$. we find the bondage number of the graphs $P_n[G]$, $n \geq 3$, where G is a graph of order m and having at most one vertex of degree $m - 1$. We also find bondage number of the graphs $P_n[G]$, $n = 4k, 4k + 1, k \geq 1$, where G is a connected graph of order $m \geq 4$ with $\Delta(G) \leq m - 2$.

REFERENCES

1. Deepak G, Indiramma M. H. and Syed Asifulla S., “Domination Number and Bondage Number of Lexicographic Product of Two Graphs”, Communicated
2. D. Bauer, F. Harary, J. Nieminen, and C. L. Suffel, “Domination alter- ation sets in graphs”, Discrete Mathematics, vol. 47 (2-3) (1983) 153-161. https://doi.org/10.1016/0012-365X(83)90085-7
3. K. Carlson, M. Develin, “On the bondage number of planar and directed graphs”, Discrete Math. 306 (2006) 820826.
4. J. F. Fink, M. S. Jacobson, L. F. Kinch, and J. Roberts, “The bondage number of a graph”, Discrete Mathematics, vol. 86 (1-3) (1990) 47-57. https://doi.org/10.1016/0012-365X(90)90348-L
5. M. Fischermann, D. Rautenbach, L. Volkmann, “Remarks on the bondage number of planar graphs”, Discrete Math. 260 (2003) 5767.
6. R. Hammack, W. Imrich, S. Klavzar, Handbook of product graphs, CRC press, 2011.
7. F. Harary, Graph Theory (Addison-Wesley, Reading, MA, 1969).
8. B. L. Hartnell, D. F. Rall, “A characterization of trees in which no edge is essential to the domination number”, Ars Combin. 33 (1992) 65-76. https://www.researchgate.net/publication/265588071
9. B. L. Hartnell, D. F. Rall, “Bounds on the bondage number of a graph”, Discrete Math., 128 (1994) 173-177. https://doi.org/10.1016/0012-365X(94)90111-2
10. B. L. Hartnell, D. F. Rall, “A bound on the size of a graph with given order and bondage number”, Discrete Math., 197/198 (1999) 409-413. https://doi.org/10.1016/S0012-365X(99)90093-6
11. T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998.
12. T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of domination in Graphs, Marcel Dekker, New York, 1998.
13. S.T. Hedetneimi, R. Laskar (Eds.), Topics in domination in graphs, Discrete Math., 86 (1990).
14. F.T. Hu, J.M. Xu, “The bondage number of mesh networks”, 7 (2012) 813-826. https://doi.org/10.1007/s11464-012-0173-x
15. L. Kang, J. Yuan, “Bondage number of planar graphs”, Discrete Math. 222 (2000) 191198.
16. Liying Kang, Moo Young Sohn, Hye Kyung Kim, “Bondage number of the discrete torus Cn × C4”, Discrete Math., 303 (2005) 80-86. https://doi.org/10.1016/j.disc.2004.12.019
17. Magda Dettlaff, Magdalena Lemanska, Ismael G. Yero, “Bondage number of grid graphs”,Discrete Applied Math., 167 (2014) 94-99. https://doi.org/10.1016/j.dam.2013.11.020
18. H. B. Walikar and B. D. Acharya, “Domination critical graphs”, National Academy Science Letters, 2 (1979) 70-72.
19. Yue-Li Wang, “On the bondage number of a graph”, Discrete Math., 159 (1996) 291- 294. https://doi.org/10.1016/0012-365X(96)00347-0

AUTHORS PROFILE

Dr. Deepak.G. obtained Ph.D at Mysore University in 2012 in ‘Graph Theory’. There are 7 International and 7 National Journal publications and has 13 years of teaching experience. Currently working as a Associate Professor in the Department of Mathematics at Sri Venkateshwara College of Engineering, Bangalore.

Mrs. Indiramma.M.H. has 8 years of teaching experience and pursuing Ph.D under the guidance of Dr.Deepak.G. Currently working as a Assistant Professor in the Department of Mathematics , Sri Venkateshwara College of Engineering , Bangalore.

Mrs. Bindu.M.G. has 10 years of teaching experience and pursuing Ph.D under the guidance of Dr.Deepak.G. Currently working as a Assistant Professor in the Department of Mathematics, Sapthagiri College of Engineering ,Bangalore.