On Laplacian and Normalized Laplacian of a Social Network

V. Yegnanarayanan, S.B. Pravallika

***Abstract*: In the task of structural identification of a network a vital tool is the underlying spectrum associated with the normalized graph Laplacian. To comprehend such spectrum, we need to determine the eigenvalues. In this paper we have found certain useful bounds involving the eigenvalues of both combinatorial Laplacian and normalized Laplacian and applied the same on a collaboration graph obtained from a social network.**

***Keywords*: Graph, Eigen Value, Spectrum, Laplacian, Normlized Laplacian, Social Network.**

I. INTRODUCTION

Graphs considered in this paper are finite, simple and undirected.

Graph theory has a crucial role in both analysis and comprehension of network structure. One can deem the components of a network as vertices and the interactions among these components as edges of the graph. By investigating the network vigorously, a new area of science, called network science, has emerged depending on graph theory. So for communication and computational networks, a fundamental model is a graph G =(V, E) consisting of a set V of vertices and a prescribed set E of unordered pairs of vertices. Spectral analysis can be employed to bring out the hidden regularity properties to give evidence on operations such as vertex duplication in the process of evolution of networks over various application domains. To find the make-up of distant stars, astronomers probe stellar spectra. Likewise, a principal goal in graph theory is to derive from its graph spectrum the main properties and to identify the underlying structure. Graph eigenvalues have prominent connections with several areas of mathematics. For instance, to differential geometry. Spectral geometry concepts and methods provide crucial insights to the investigation of graph eigenvalues. Spectral methods that are algebraic in nature also contributed significantly to the creation ofextremal examples and constructions. Also it is well known that spectral graph theory played a significant role in chemistry. Stability of molecules have close association with eigenvalues.

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Graph spectra occurs in several tasks of theoretical physics and quantum mechanics, for instance, in Hamiltonian systems for energy minimization. Lately expander graphs and their eigenvalues are determined to address challengesoccurring in communication networks.

To probe real systems, networks are designed. As per the nature and area of the system, networks are divided into four types. They are biological (protein-protein interactions, gene regulation food web etc.,), technical (electronic, circuits, power grids air/train/bus routes, internet etc.,), information(citation, www etc.,) and social(scientific co-authorship, friendships, e-mail contacts etc.,). In this paper, we consider social networks. In particular, the scientific collaboration networks. Here scientists are denoted by vertices and two vertices are joined by an edge if the corresponding scientists possess a joint research paper.

While applying graph theory in network analysis we have to keep in mind that that the number of non-isomorphic graphs is confusingly high. So it is next to impossible to list all possible distinct graphs unless the cardinality of the vertex set is small. Moreover, visual analysis is infeasible on a graph with huge number of vertices and edges as it appears too convoluted to provide a transparent structure.

Lately real systems produce huge data sets which in turn brings new directions to existing scientific issues.

1. Deeming a specific structure, which features within a class are universal and which ones are shared by other structures?
2. Deeming a complex structure, how to obtain a representation that marks qualitative features?
3. Is it possible to identify the domain resting on unique qualitative features associated with the graph?

To find answers, we propose a class of graph invariants that provide both qualitative characterization and can be visually analyzed and compared. This class is what is called the spectrum of the graph Laplacian.

A matrix that plays a crucial role in this paper is the Laplacian of the graph. This matrix has close connections to a variety of graph-theoretic notions. One can relate linear-algebraic properties of the Laplacian to the behavior of spanning trees of the underlying graph. Further the computation of spectrum or the eigen values of the matrix of a graph are crucial in the investigation of discrete isoperimetric inequalities and the like. For a detailed study on the role of Isoperimetric inequalities within the domain of Riemannian geometry one can refer to [1,2]). An investigation of these isoperimetric inequalities in the context of graphs with symmetry are done in [3].Kirchhoff 'sMatrix-tree theorem in [4] mentions that the determinant of a principal minor of the combinatorial Laplacian yields the number of spanning trees in a graph.

Moreover,Donath et al. [5, 6] first used spectral techniques due to its global appeal on optimization problems in preference to local search approaches and then by many researchers [7-12].For more on various networks and the role of spectrum on some of these networks one can also see [13-35]. Also there is a connection between graph cuts and eigenvalues of a matrix called Laplacian.

One can deduce certain graph properties from eigenvalue distributions. For this it is apt to considerthenormalizedLaplacian$\mathscr{L}$, $\mathscr{L}= D^{-1/2}LD^{-1/2}$instead of combinatorial Laplacian L. We do not allow isolated vertices to ensure that D is invertible, Up to a scale factor for regular graphs $\mathscr{L}$ and L are the same. But, for general graphs, it is convenient to use $\mathscr{L}$. The eigenvalues of$\mathscr{L}$ and L are denoted respectively by $0=\sigma_0\leq\sigma_1\leq\ldots\leq\sigma_{n-1}and0=\lambda_0\leq\lambda_1\leq\ldots\leq\lambda_{n-1}$.

It is interesting to note that for the complete graph K_p on p vertices, the eigenvalues are 0 and p/(p − 1) with multiplicity p – 1; For the complete bipartite graph $K_{r,s}$, on r + s vertices, the eigenvalues are 0, 1 with multiplicity r + s − 2 and 2; For the star S_p on p vertices, the eigenvalues are 0, 1 with multiplicity p − 2, and 2; For the path P_p on p vertices, the eigenvalues are 1 − cos πk p−1 for k = 0, 1, . . . , p – 1; For the cycle C_p on p vertices, the eigenvalues are 1 − cos 2πk p for k = 0, . . . , p − 1. For the p-cube Q_p on 2^p vertices, the eigenvalues are 2k /p with multiplicity p!/k!(p-k)! for k = 0, . . . , p.

Main task of spectral theory revolves around the derivation of bounds on the eigenvalues distributions. The issue of reducing the range of the eigenvalues results in open-ended challenge for certain classes of graphs.

The clustering coefficient C of a graph G is defined as, C = (3 × number of triangles)/total number of connected triples of vertices. Note that $0 \leq C \leq 1$. Another variation for C is C = (1/|V(G)|) Σ C_i where C_i = number of triangles connected to vertex i/ number of triangles centered on vertex i = $2E_i$ / $p_i(p_i$-1) where E_i is the number of edges between the neighbors of vertex i.

II. LAPLACIAN AND NORMALIZED LAPLACIAN

To obtain the number of spanning trees of a graph through the evolution of the determinant of a matrix , we use Matrix-Tree Theorem. A tree is a connected graph that includes no cycles. A spanning subgraph H of a graph G is the one with V(H) = V(G) and every edge e ∈ E(H) belongs to E(G). As we can choose any subset of the edges of G to be E(H) we infer that if |E(G)| = q then the number of spanning subgraphs of G is 2^q . A spanning tree is a spanning subgraph and a tree. Number of spanning subgraphs in a graph is a pertinent graph structural parameter and is denoted by τ(G).
Cauchy-Binet Theorem provides a nice generalization of a vital property concerning determinants that if A and B are n×n matrices, then |AB|=|A||B| to the case then when A and B are rectangular matrices. Interestingly it turns that |AB| is a square matrix.

Suppose that A = $[u_{ij}]$ is a p×qmatrix , with $1 \leq i \leq p$, $1 \leq j \leq q$ and $p \leq q$. If S is a p-element subset of {1,2,...,q} , then we denote by A[S] the p×psubmatrix of A formed by the columns indexed by the elements of S . That is , if the elements of S are taken as $j_1< j_2< \ldots<j_p$, then A[S] = $(a_{i\,,\,jr})$ where $1 \leq i \leq p$ and $1 \leq r \leq p$.

Illustration 1 : if A = $\begin{bmatrix} 10 & 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 & 25 \\ 31 & 32 & 33 & 34 & 35 \end{bmatrix}$ and S = {10,12,14} then A[S] = $\begin{bmatrix} 10 & 12 & 14 \\ 21 & 23 & 25 \\ 31 & 33 & 35 \end{bmatrix}$

Likewise, if B = (b_{ij}) is a q×p matrix with $1 \leq i \leq q$, $1 \leq j \leq p$ and $p \leq q$. Let S be a p-element subset of {1,2,…,p} . Then B[S] is an p×p matrix formed by the rows of B indexed by S. Observe that $A^T[S] = A[S]^T$, where T stands for the transpose of a matrix.

Cachy-BinetTheorem :
Let A = (a_{ij}) be an p×qmatrix, with $1 \leq i \leq p$ and $1 \leq j \leq q$. Let B = (b_{ij}) be anq×p matrix with $1 \leq i \leq q$ and $1 \leq j \leq p$. If p>q then |AB| = 0. If $p \leq q$, then |AB| = ∑ |A[S]| |B[S]| where S ranges over all p-element subsets of {1,2,……,p}.

Let G be a graph with |V(G)| = p where V(G) ={v_1,…,v_p}. An adjacency matrix A(G) of G is an p×pmatrix over ,the field of complex numbers where a typical entry a_{ij} stands for the number of edges is evident to v_i and v_j. Observe that A(G) is symmetric and hence its eigenvalues are real.

The Laplacian matrix L(G) of the graph G is an p×p matrix where $L_{ij} = -r_{ij}$ if $i \neq j$ and there are r_{ij}edges between v_i and $v_j$and L_{ij}= deg(v_i), if i=j. As G is simple, r_{ij} = 1 always. Moreover one can also define L(G) as L(G) = D(G)-A(G) where D(G) is the diagonal matrix with each diagonal entry to be the degree of the respective vertex. Here again one can observe that L(G) is symmetric.

The incidence matrix M(G) of G is an p×q matrix , where q = |E(G)| where M_{ij} = 1 if the edge $e_j$emanates from v_i , M_{ij}= -1 if the edge $e_j$ends with v_i , and M_{ij}= 0 otherwise. Observe that each column of M(G) has (q-2) 0's , one 1 and one -1 and so the sum of the entries in every column is 0. This means all the rows sum to 0 vector and hence exhibits linear dependence relation with rank (M(G))<p. Note that for u_i,u_j∈ V(G) we have $M(G)M(G)^T = \sum M_{ik}M_{jk}$. If $i \neq j$ then e_k∈E(G) joins v_i and v_j as $M_{ik}M_{jk}\neq 0$ in order . So one of $M_{ik}$and M_{jk}will be 1 and the other -1. That is their product $M_{ik}M_{jk}$is always -1. If i=j then $M_{ik}M_{jk}$= 1 if $e_k$is an edge with $v_i$as its one vertex and 0 otherwise. This means $(M(G)M(G)^T)_{ij}$ = deg(u_i) and so $L(G) = M(G)M(G)^T$.

Let G be a connected graph. Let M_o(G) be a submatrix of M(G) with its last row dropped. Then M_o(G) has order (p-1)×q. Observe that the number of rows is equal to the number of edges in a spanning tree of G. Let S be a set of (p-1) edges of G. If it is not the set of edges of a spanning tree, then some subset R of S becomes the edges of a cycle C in G. Let M_o[R] be a submatrix got from M_o[S] by considering the columns indexed by edges in R. Let e_1', e_2',…, e_t' be the edges of R. Now multiply the columns of M_o[R] indexed by f_j by 1 if we traverse f_i while going around C and multiply by -1 otherwise. This operation results in multiplication of |M_o[R]| by ±1. Note that every row of this transformed M_o[R] has sum of its elements 0. As a result the sum of all the elements is 0. So in M_o[S] we have a set of columns for which a linear combination with coefficient ±1 is 0. That is column of M_o[S] are linearly dependent, and |M_o[S]| = 0.

If S is the set of edges of a spanning tree T, then let f be an edge of T connected to u_p. Here u_p is the vertex that indexes the bottom row of M. More specifically, M_o is derived by deleting this row.

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The column of $M_o[S]$ indexed by f contains exactly one non-zero entry, which is ±1 . Delete from $M_o[S]$ the row and column containing the non-zero entry of column f. It results in a submatrix M_0' of order (p-2)×(p-2). Observe that $| M_o[S]| = \pm| M_o'|$. Let T^* be the tree got from T by contracting the edge f to a single vertex v. That is, u_p and the remaining vertex of the column of f are merged to form v. Then M_o' is the matrix got from M(T') by removing the row indexed by v. Now by adopting the induction argument on p, one can get $|M_o'| = \pm 1$. So $|M_o[S]| = \pm 1$. Hence

Theorem 1. If a connected graph G has a set S of (p-1) edges and the edges of S are not the edges of a spanning tree, then $|M_o[S]| = 0$. Else $|M_o[S]| = \pm 1$.

Now as $L(G) = M(G)M(G)^T$ it is clear that $L_0(G) = M_0(G)M_{0(}G)^T$. By invoking Cauchy-Binet theorem, $|L_o(G)| = \sum |M_o[S]|\ |M_o{}^T[S]|$ where S ranges over all (p-1) element subsets of {1,…,q}. As $A^T[S] = A[S]^T$, we get $|L_o(G)| = \sum |M_o[S]|^2$. But by Theorem-1, $|M_o[S]| = \pm 1$ if S forms the set of edges of a spanning tree of G , and 0 otherwise. So the term indexed by S in the sum is 1 if S is the set of edges of a spanning tree of G and is 0 otherwise. So

Theorem 2.(**Matrix-Tree Theorem**)

Let G be a connected graph with L(G). Let L_o denote L with last row and column removed. Then $|L_o(G)| = \tau(G)$.

We have provided in the above discussion a brief outline of the proof of well known Matrix-Tree Theorem for sake of completeness.

Theorem 3. Let P be a p×p matrix with all its entries from the set of all real numbers. Suppose that the sum of the entries in each row and column is 0. Let P_o be a submatrix of P formed by deleting any row and any column. Then the coefficient of x in the |P-xI| of P is $-p|P_0|$. Further |P-xI| is constant term free.

Proof. As |P|= 0 due to the fact that the rows of P sum to 0, we get the constant term of |P-xI|= |P| = 0 and hence it is constant term free. We prove this first for the case where we drop the last row and last column. Similar logic works fine when we drop any row and any column. Perform sum operation on all the rows of P-xI except the last row. This will not alter the determinant but will change the entries of the last row all to -x. If we factor out -x from the last row, we get a matrix Q(x) fulfilling |P-xI| = -x|Q(x)|. So the coefficient of x in |P-xI| is -|Q(0)|. Now perform the sum operation on all the columns of Q(0) except the last column, This will not alter |Q(0)|. As the columns of P sum to 0, the last column of Q(0) is the vector $[0,0,\ldots,0,p]^T$. Now expand the determinant by the last column to see $|Q(0)| = +p|P_o|$. Therefore the coefficient of x in |p-xI|= $-p|P_o|$.

As interesting deduction of the Theorem 3 is that if $\lambda_1,\ldots,\lambda_{p-1}$, $\lambda_p = 0$ are the eigen values of L(G) then $\tau(G) = (1/p)(\ \Pi\ \lambda_i)i = 1$ to p-1. This is because, $|L-xI| = (\lambda_1-x)\ldots(\lambda_{p-1}-x)(\lambda_p-x) = -(\lambda_1-x)\ldots(\lambda_{p-1}-x)x$. Hence the coefficient of x is $-\lambda_1 \lambda_2\ldots\lambda_{p-1}$. By Theorem 3, we get $-\lambda_1 \lambda_2\ldots\ldots\ldots\lambda_{p-1} = p|L_o|$. By Theorem 2, we get $|L_o| = \tau(G)$ and hence $\tau(G) = (1/p)(\ \Pi\ \lambda_i)$.

Theorem 4. If {deg(v)},v ∈ V(G) is the degree sequence of a graph G then $\tau(G)=(\Pi d_v/\sum d_v)\Pi\sigma_i$ where σ_i's are non-zero eigen values of $\mathscr{L}$ or $\tau(G) = (\deg(v_1)\ldots\deg(v_{|V(G)|})/\deg(v_1)+\ldots+\deg(v_{|V(G)|}))(\sigma_1\sigma_2\ldots\sigma_i)$.

Proof. Look at the first degree terms in the expansion of $|\mathscr{L}-xI|$. One can see that $\sum|\mathscr{L}_v|= \sigma_1\sigma_2\ldots\sigma_i$ where each $\sigma_i\neq 0$ and $\mathscr{L}_v$ is obtained by omitting the row and column corresponding to v. From Theorem 2, $|\mathscr{L}_v| = |L_0|/\Pi\deg(u)= \tau(G)/\Pi\deg(u)$, u≠v. Hence the result.

Suppose that X ⊆ V (G) be a subset of vertex set of a graph G. Set $V^0(X) =\deg(v_1)+\ldots+ \deg(v_{|(X)|})$ and $V^0(G) =\deg(v_1) +\ldots+ \deg(v_{|V(G)|})$. Let $E^0 = E(X, Y)$ be the set of all those edges e=(x, y) of G with one end x ∈ X and the other end y ∈ Y.

Theorem 5. Suppose that X, Y⊆ V (G) are any two subset of the vertex set V of a graph G. Then $|E^0-(V^0(X)V^0(Y)/V^0(G)| \leq \sigma^*((V^0(X) V^0(X^c) V^0(Y) V^0(Y^c))^{1/2}/ V^0(G))$ where $X^c=V-X$, $Y^c=V-Y$ and $\sigma^* = \max \{|1- \sigma_i|\}$ where σ_i are non-zero eigenvalue of $\mathscr{L}$.

Proof. For $\mathscr{L}$, let Ψ_i be the orthonormal eigen function of $\mathscr{L}$ annexed with non-zero σ_i and $D^{1/2}\chi_x= \sum_i\alpha_i\Psi_i$, $D^{1/2}\chi_y=\sum_i\beta_i\Psi_i$ .Then One can see that $\Psi_o(v)= (\deg(v) / V^0(G))^{1/2}$, and $\alpha_0\beta_0 = (V^0(X) V^0(Y)/ V^0(G))$. Then $|E^0- (V^0(X) V^0(Y)/ V^0(G))|= |E^0-\alpha_0\beta_0|=\sum_{i\neq 0} \alpha_i\beta_i-(1-\sigma_i) \leq\sigma^*\sum_{i\neq 0} |\alpha_i -\beta_i|\leq \sigma^*\sum_{i\neq 0} |\alpha_i\beta_i| \leq \sigma^*(\sum_{i\neq 0} \alpha_i^2 \sum_{i\neq 0} \beta_i^2)= \sigma^*(V^0(X) V^0(X^c) V^0(Y) V^0(Y^c) / V^0(G))$ by Cauchy-Schwartz inequality. Here χ_x stands for the characteristic function of X and Y.

Theorem 6. If G is a graph with |V (G)|= p, $d=\sum (\deg(v))/p$ where v ∈V(G)and the absolute value of d and the non-zero Eigen value λ_i are at most η for some $\eta\in$ R then $|Z| = |E^0-(d/p)|X||Y|+d |X\cap Y| -V^0(X\cap Y)|$ is bounded above by $(\eta/p)\sqrt{((p-|X|)|X||Y|(p-|Y|))}$ for any two subset X,Y of V(G), where $Z= E^0- (d/p)|X||Y|+ d |X \cap Y| -V^0(X \cap Y)$.

Proof. For i=0,1,…,p-1, let Ψ_i be the orthonormal eigenvectors of L. Then $\Psi_0=1/\sqrt{(p)}$ where 1=(1,1,...,1). Set $\chi_x=\sum\alpha_i\Psi_i$ and $\chi_y=\sum\beta_i\Psi_i$ and I be the identity of matrix. Note that $E^0+d|X\cap Y|=<\chi_x,(A+\alpha I)\chi_y>$ where < . > stands for the inner product. As L=D-A we see that $<\chi_x,(A+\alpha I) \chi_y> = <\chi_x,(D-L)+dI)\chi_y> = <\chi_x,(D-L) \chi_y> + dI<\chi_x, \chi_y> =V^0(X\cap Y)-\sum\lambda_I\alpha_i\beta_i+d\sum\alpha_i\beta_I = V^0(X\cap Y)- \sum (d-\lambda_i)\alpha_i\beta_i+ d\alpha_0\beta_0$. As $\alpha_0=< \chi_x,\Psi_0> =|X|/\sqrt{(p)}$ and $<\chi_y, \Psi_0>=|Y|/\sqrt{(p)}$ we get $E^0+d|X\cap Y|= V^0(X\cap Y)- \sum (d-\lambda_i)\alpha_i\beta_I + d/p|X||Y|$. So $|Z|=|E^0-(d/p)|X||Y|+d|X\cap Y|-V^0(X\cap Y)|\leq\sum(d-\lambda_i)\alpha_i\beta_i\leq\eta\sum i_{\neq 0} \alpha_i\beta_I$ where $i \neq 0$. By Cauchy-Schwartz inequality we have $\sum\alpha_i\beta_I \leq\sqrt{(\sum\alpha_i^2 \sum\beta_i^2)} \leq\sqrt{((||\chi_x||^2-\alpha_0^2)(||\chi_y||^2-\beta_0^2))}$. Hence $|Z|\leq(\eta/p)\sqrt{((|X|(p-|X|)|Y|(p-|Y|))}$.

Corollary 6.1. $|E^0-(d/p)|X||Y||\leq \eta/p\sqrt{(|X|(p-|X|)|Y|(p-|Y|)}$

Corollary 6.2. If $|X|\geq \eta$ p/d and $|Y|\geq \eta p/d$ in Theorem 6 then there is at least one edge between a vertex of x and a vertex of Y.

Proof. This is because, if $E^0=0$ then we have $|(d/p)|X||Y||\leq \eta/p\sqrt{(|X||Y|)}\sqrt{(p-|X|)(p-|Y|)}$. So $d\sqrt{|X||Y|}\leq\eta\sqrt{((p-|X|)(p-|Y|))}$. But our hypothesis $|X||Y|\geq\eta^2p^2/d^2$ and hence $\sqrt{(|X||Y|)}\geq\eta p/d$ Hence $d\sqrt{(|X||Y|)} \geq \eta p$. This implies $p \leq\sqrt{((p-|X|)(p-|Y))}$, a contradiction.

By taking Y=p-|x| in corollary 6.1 we derive

Corollary 6.3. $(d-\eta/p)|X|X^c| \leq E^0\leq((\alpha+\eta/p)|X||X^c|$

Next by taking Y ={ w } in corollary 6.1 we derive

Corollary 6.4. $(p/p-1)(d-\eta)\leq \deg(v) \leq (p/p-1)(d+\eta)<d+\eta$ for all v ∈ V(G)

Corollary 6.5. If we let $E^{00}=E(X,X)$ then $|2E^{00}-d|X|(|X|-1)/p)|\leq(2\eta/p)|X|(p-|X|/2)$

Corollary 6.6. **If** S is an independent set in G then $|S|\leq\eta p/d+1$.

III. ROLF NEVANLINNA PRIZE WINNER'S COLLABORATION GRAPH (RNPCG)

By G^* we mean RNPCG for the prize winners between1982 to 2014.

For the method of construction of the graph one can refer to [36-40].First note that the largest Erdos number among all the prize winners and their collaborators is only 3. This means there are four levels upto which the collaboration stretches starting from Erdos at level 0 to level 3. By Level i we mean collaborators with Erdos number i. A procedure for determining the vertex and edge set of G* are discussed in [36-40]. We verify the results obtained in Section 2 for the collaboration graph G*. The vertex set of G*is V (G*) = $\{w_1, w_2, \ldots, w_{80},\}$where w_i's are the following:

w_1 = Paul Erd¨os w_{41} = Amit Chakrabartiw_2 = Maria MargaratKlawe w_{42} = NisheethK.Vishnoiw_3 = SiemionFajtlowicz
w_{43} = Nikhil Bansal w_4 = Robert Robinson
w_{44} = Vitaly Feldman w_5 = George Gunthar Lorentz w_{45} = Per Austrinw_6 = EndreSzemeredi w_{46} = OdedRegevw_7 = Laszlo Lovasz w_{47} = ParikshitGopalanw_8 = Nathan Linial
w_{48} = Aranyak Mehta w_9 = AlonNoga
w_{49} = Ryan O' Donnell w_{10} = Boris Aronov
w_{50} = Mulisafraw_{11} = Andrej Ehrenfeucht
w_{51} = William Frank Perkins w_{12} = Mark Jerrum
w_{52} = Jonas Holmerinw_{13} = Alok Aggarwal
w_{53} = Yaoyunshiw_{14} = Robert EndreTarjanw_{54} = Nikhil R. Devanurw_{15} = Leslie Valiant w_{55} = IritDinurw_{16} = A.A. Razborovw_{56} = Alexandra Kollaw_{17} = AviWigdersonw_{57} = PopatPreyasw_{18} = Peter W. Shorw_{58} = Saket Rishi w_{19} = Madhu Sudan
w_{59} = MadhurTulsianiw_{20} = Jon Kleinberg
w_{60} = Yi Wu w_{21} = Mario Szegedyw_{61} = Ashok Kumar Ponnuswamiw_{22} = Lance J. Fortnow w_{62} = Shlomo Moran w_{23} = Daniel Spielman
w_{63} = SvanteJansonw_{24} = SubhashKhotw_{64} = Joel H. Spencer
w_{25} = Sanjeev Arora w_{65} = David M. Avis w_{26} = Richard J. Lipton
w_{66} = Andrew chi chihyaow_{27} = Johan T. Hastadw_{67} = Robert D. Kleinberg w_{28} = VenkatesanGuruswami w_{68} = MiklosAjtaiw_{29} = T.S. Jayramw_{69} = Prasad Tetaliw_{30} = Howard J. Karloff w_{70} = Santosh S. Vempalaw_{31} = Guy Kindler w_{71} = Michael Klugermanw_{32} = S. Ravi Kumar
w_{72} = Alexander C. Russell w_{33} = EvangelosMarkakis w_{73} = Andrew M. Odlyzkow_{34} = Mikhail Alekhnovich w_{74} = Michael E. Saks w_{35} = Yuval Rabaniw_{75} = Bella Bollobasw_{36} = Venkatesh Raman
w_{76} = Leonard J. Schulman w_{37} = Dana Moshkovitzw_{77} = Vijay V. Vaziraniw_{38} = ElchananMossel
w_{78} = Dieter Kratschw_{39} = AssafNaor
w_{79} = AndrisAmbainisw_{40} = David steurer
w_{80} = David Zuckerman

The edge set of G* is denoted by E(G*) and E(G*) = $\{f_1, f_2, \ldots, f_{330}\}$. The fi's are described as follows:

$f_1 = (w_1, w_2)$ $f_2 = (w_1, w_3)$ $f_3 = (w_1, w_4)$ $f_4 = (w_1, w_5)$ $f_5 = (w_1, w_6)$ $f_6 = (w_1, w_7)$ $f_7 = (w_1, w_8)$ $f_8 = (w_1, w_9)$ $f_9 = (w_1, w_{10})$ $f_{10} = (w_1, w_{21})$ $f_{11}=(w_1,w_{62})$ $f_{12}= (w_1, w_{63})$ $f_{13} = (w_1, w_{64})$ $f_{14} = (w_1,w_{65})$ $f_{15} = (w_1, w_{68})$ $f_{16}= (w_1, w_{69})$ $f_{17} = (w_1, w_{71})$ $f_{18} = (w_1, w_{73})$
$f_{19} = (w_1, w_{74})$ $f_{20} = (w_1, w_{75})$ $f_{21} = (w_1, w_{76})$
$f_{22} = (w_1, w_{78})$ $f_{23} = (w_2, w_8)$ $f_{24} = (w_2, w_{13})$
$f_{25} = (w_2, w_{14})$ $f_{26} = (w_2, w_{17})$ $f_{27} = (w_2, w_{18})$
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Method of drawing the collaboration graph G* is described in [36-40]. As G* is fairly a large graph with 80 vertices and 330 edges for manual drawing we employed Pajek program to visualize it. In Windows operating system Pajek is a amenable program for the twin purpose of visualizing and analyzing G*. For the latest version of Pajek one can refer to http://pajek.imfm.si/doku.php?id=download. For the entire procedure of downloading, installing and using the pajek program one can refer to http://pajek.imfm.si/doku.php?id=download.Figure 1 The 1982-2014 RNP collaboration graph G*

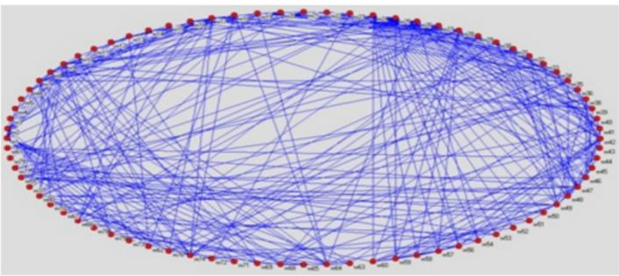

Figure 1. TheRNP Collaboration Graph G*

IV. DISCUSSION

The eigenvalues of the matrix L(G*) of the collaboration graph G* is computed through MATLAB. The product of eigenvalues of L(G*) is equal to 1.41647e+63 and the number of spanning trees of G* 1.77059e+61. Then we have computed the number of spanning trees of G* through the other formula namely $\det(M_0(G^*)M_0(G^*)^T)$ and it very well gives the same result of 1.77059e+61 satisfying Theorem 4. Similarly we have computed the eigenvalues of $\mathcal{L}$ matrix of G*. The number of spanning trees is calculated as 1.77055e+61 and is equal to $\tau(G)=(\Pi d_v/\sum d_v)\Pi\sigma_i$and satisfies Theorem 4. Then for the purpose of 1) to display adjacency matrix, incidence matrix, diagonal matrix and transpose of incidence matrix 2) calculating the L and $\mathcal{L}$ matrices, 3) product $M(G^*)M(G^*)^T$, 4) find $M_0(G^*)$, $M_0(G^*)^T$ and product $M_0(G^*)M_0(G^*)^T$, 4) product of the eigenvalues of the matrix L, 5) to check the validity of Theorems 4 to 6 and their corollaries we have made use of the C++ coding.

V. FUTURE WORK AND CONCLUSIONS

Laplacian of a complete graph K_p is special as each vertex is joined to all other vertices. So the diagonal entries of the Laplacian are p-1. Also the adjacency matrix of K_p has 1 on each entry except the diagonal entries. Therefore Laplacian of $K_p$is $(L_{complete})_{ij}$= p-1 if i=j and -1 otherwise. Observe that the L matrix of a graph G is symmetric if and only if G is undirected. In view of this the RNP collaboration graph G* is symmetric as it is undirected. Note the number of corrected components in a graph is equal to the multiplicity of the eigenvalue 0 of the L matrix of the corresponding graph. To see this, suppose that $\{G_i\}_{1\le i\le k}$ are the k-connected components of G. Let $\chi_{i\,j}=1$ be the characteristic vector of G_i. That is χ_{ij}= 1 if $j\in G_i$ and 0 otherwise. Then each 1_i is an eigenvector of eigenvalue 0 and all such vectors are orthogonal. Conversely if $y \in R^p-\{0\}$, is an eigenvector corresponding to eigenvalue 0, then $y^TAy = \sum (y_i-y_j)^2 = 0$. This means the value of y is same over any edge and it splits the vector into k characteristic vectors of the connected components. So there is at least k vectors to span the respective eigenspan. Hence G is connected. The RNP collaboration graph G* has only one eigenvalue with value 0 and therefore it has only one component. Hence it is connected.

We know that the smallest eigenvalue is λ_0= 0. If G is connected the other λ_i> 0 for I > 0. Arrange all of them in increasing order: $0=\lambda_0<\lambda_1\le\lambda_2\le\ldots\le\lambda_{p-1.}$ It is clear that $\lambda_{p-1}\le 2$ and $\lambda_{p-1} = 2$ if and only if the respective graph is bipartite. So the difference $2-\lambda_{p-1}$ provide an idea regarding how a given graph differs from a bipartite graph. Also note that if λ is an eigenvalue then $2-\lambda$ is also an eigenvalue. If G is complete then $\lambda_1=\lambda_2=\ldots=\lambda_{p-1}=p/(p-1)$ and hence $0\le\lambda_1\le p/(p-1)\le\lambda_{p-1}\le 2$. If G is not complete then $\lambda_1\le 1$. Further, λ_1, the second smallest eigenvalue is a powerful graph invariant. It reveals how hard it is cut the graph into two components $V_1(G),V_2(G)$ such that $V_1\cap V_2=\emptyset$. If $X\subseteq V(G)$ and $\partial X=\{(u,v) \in E(G) : u\in X , v \in V(G)-X \}$. Then the Cheeger constant of G, h(G) is defined as: $h(G) = \min \{|\partial X|/|X| : X\subseteq V(G) , 0<|X|\le\frac{1}{2}|V(G)|\}$. It is known that h(G)>0 if and only if G is connected.

The fact that h(G) is small and positive indicates a scenario where there are two sets of vertices with few edges between them. So h(G) is large and positive stands for the case when there are many edges between two partitions of the vertex set of G. In the RNP collaboration graph G* of a social network , the h value is large and that is what is desirable. Further a relation between h and λ_1 is that $½h^2(G) \leq \lambda_1 \leq h(G)$. Note that the RNP collaboration graph G* satisfies this.

The spectrum of the normalized graph Laplacian gives pertinent invariants of the corresponding graph G and helps in bringing out the qualitative properties. It is possible to recover the graph from its spectrum as in the case of combinatorial Laplacian, up to isospectral graphs. One can see [32] for a systematic treatment on this aspect. We intend to do this for our collaboration graph G^* in future elsewhere. Also we will try to extend a heuristic algorithm for the algebraic Laplacian L to the normalized Laplacian by perusing [33].

We propose to develop a scheme for the classification of networks depending on qualitative properties observed through the spectrum of the Laplacian of the graph underlying the network. We also intend to give new constructions through various graph operations related to the evolution of the considered social network and generate specific eigenvalues to explain graph formation that imposes characteristic traces in the spectrum. We will investigate the spectrum of a graph constructed from actual data to comprehend the evolutionary processes of the social network.

The multiplicity of the eigenvalue 1 is crucial as several spectral plots of social networks have a very sharp peak around 1. We will investigate how multiplicity changes according to several graph operations, such asnode duplication, random rewiring, random edge deletion, motif duplication, graph joining and splitting that are evolutionary in nature. We also wish to probe the eigenvalues ½ and 3/2 (that occur more frequently in real networks) of various graphs that gets generated due to graph operations on the original graph. Understanding the structure of those graphs that are relatively small in size will help us to understand the complicated structure of the original graph.We further wish to investigate by applying the nodal domain theorem on a graph to find the relationship between the multiplicity of a certain eigenvalue and the number occurrences of a certain motif in the graph.

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