To Asymptotic of the Solution of the Heat Conduction Problem with Double Nonlinearity, Variable Density, Absorption at a Critical Parameter

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Abstract: In this work we have established asymptotic behavior \((f(t) \to \infty)\) of solutions of the Cauchy problem for a nonlinear heat equation with variable density describing the diffusion of heat with nonlinear heat absorption at the critical exponent of the parameter \(\beta\). For numerical computations as an initial approximation we used founded the long time asymptotic of the solution. Numerical experiments and visualization were carried for one and two dimensional case.

Keywords: critical value of parameter, heat conduction problem, maximum principle, numerical computation, variable density.

I. INTRODUCTION

As is well known for the numerical computation of a nonlinear problem, the choice of the initial approximation is essential, which preserves the properties of the final speed of propagation, spatial localization, bounded and blow-up solutions, which guarantees convergence with a given accuracy to the solution of the problem with minimum number of iterations.

It is very important to establish the values of numerical parameters at which the nature of the asymptotic behavior of the solution will change. Such values of numerical parameters are called critical or critical values of the Fujita type. He first established this for the semi-linear heat equation [1]. At the critical parameters we can observe new effects such as infinite, localization and others.

In the area \(Q = \{t, x\} : t > 0, x \in \mathbb{R}^n\) the Cauchy problem which given below

\[
\frac{\partial u}{\partial t} = \nabla \left( |x|^m u^{m-1} |\nabla u|^p \right) - u^\beta
\]

\(u(0, x) = \varphi_0(x), \quad x \in \mathbb{R}^n \) \hspace{1cm} (1)

\(t\) and \(x\) are, respectively, the temporal and spatial coordinates where \(m \geq 1, k \geq 1, p \geq 2, n \geq 1\) given numerical parameters, characterizing the nonlinear medium \(\nu(x) = \text{grad}(\cdot)\).

\(\beta = k(p-2)+m\frac{p}{s}, \quad s = pN/(p-n), p > n \quad (\beta_c - \text{critical value, } N \text{-size of dimension})\). In case \(k=1, n=0\) critical value of parameter \(\beta\) was found in [2].

Problem (1) - (2) is the basis for modeling various processes of nonlinear heat diffusion, magnetic hydrodynamics, gas and liquid filtration, oil and gas.

A lot of works studied properties of solutions of problem with critical value of parameter \(\beta\) and were established asymptotic behavior for \(t \to \infty\) (see [1]-[5]). The long time asymptotic of solutions has been established for the critical exponent of parameter \(\beta\) for problem (1)-(2) in case \(k=1, m=1, n=0\). They proved that for problem (1) - (2) in case \(p>2, n=0, m=1\). They considered following semi-linear parabolic equation

\[
\frac{\partial u}{\partial t} - \Delta u + u^\beta = 0, t > 0
\]

\(u(x, 0) = u_0(x) \geq 0 \hspace{1cm} (3)

\(\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \hspace{1cm} (4)

\hspace{1cm} \beta = 1 + \frac{2}{N} \)

The solution of problem (3) - (4) is “infinity” energy. The starting data is

\(u_0(x) = o\{\exp(-\gamma |x|^2)\}, \quad x \to +\infty \hspace{1cm} \gamma > 0 \)

They proved that for problem (3) - (4) the asymptotic of the solutions for \(t \to \infty\) is the following approximate self-similar solution

\[
u(t, x) = \left[\left( T + t \right) \ln(T + t) \right]^{-\frac{N}{2}} \rho \left( \frac{x}{(T + t)^{\frac{1}{2}}} \right) \hspace{1cm} (5)

\]

For \(g\), function upper and lower bounds were obtained

\[A \exp \left( -\frac{|\xi|^2}{4} \right) \leq g(\xi) \leq H \exp \left( -\frac{|\xi|^2}{4} \right) \]

\(|\xi|^2 = \frac{|x|^2}{T + t} \)

where \(A, H\) - constants.
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For $\beta \neq 1 + \frac{2}{N}$, the approximate self-similar differs from (5), which means that for critical values the asymptotic of the solutions will change for $t \to \infty$.

In this work we have established asymptotic behavior for $t \to \infty$ of solutions of the problem (1)-(2) for a nonlinear heat equation with variable density, describing the diffusion of heat with nonlinear heat absorption at the critical exponent of the parameter $\beta$.

II. ASYMPTOTIC OF THE SOLUTION

Based on the method [2], the solution to the problem (1) - (2) will be found in the form which given below

$$u(t, x) = \tilde{u}(t) \phi(x, \tau(t))$$

where $\tilde{u}(t) = (t \ln t)^{-\frac{1}{p-n}}$

Put (6) in (1) and select $\tau(t)$

$$\tau(t) = \int (t \ln t)^{-\frac{m+k(p-2)+1}{p-1}} dt$$

For $\phi(\tau(t), x)$ we get following equation

$$\frac{d\phi}{d\tau} + \nabla(x) \nabla \phi | \nabla \phi |^{p-2} \nabla \phi = -u - \frac{m+k(p-2)+1}{p-1} \phi$$

We note that for large $t$ by the Hardy theorem on the behavior of the integral [2], the function $\tau(t)$ has the following asymptotic

$$\tau(t) \sim t^{\alpha_1} \ln^{\alpha_2} t, \ t \to \infty$$

Where

$$\alpha_1 = \frac{k(p-2) + m - 1}{(p+k(p-2)+m)s}, \ \alpha_2 = \frac{(p-2)k + m - 1}{(p+k(p-2)+m)s} + 1$$

Now put $\phi = f(\tilde{\xi})$ where

$$\tilde{\xi} = \phi(x)[\tau(t)]^{-\frac{1}{p-n}}, \phi(x) = \frac{p}{p-n} |x|^{\frac{p-n}{p}}, \ n < p$$

in (7) we get an approximately self-similar equation:

$$\frac{\tilde{\xi}^{1-N}}{d} \frac{d}{d\tilde{\xi}} (\tilde{\xi}^{N-1}) - f(\tilde{\xi}) + \frac{d}{d\tilde{\xi}} \frac{\tilde{\xi}}{p} \frac{df}{d\tilde{\xi}} = \frac{df}{d\tilde{\xi}}$$

(8)

We take as a generalized solution of the equation (8) the following function

$$\bar{f}(\tilde{\xi}) = \left(a - b\tilde{\xi}^{\frac{p}{p-n}}\right)^{\frac{p-1}{(m+k(p-2)+1)/m-1}}$$

(9)

$b = (k(p-2) + m - 1) / (k(p-2)^{-\frac{1}{p-1}}), \ m + k(p-2) - 1 > 0$

After putting (9) into (8) a self-similar equation transforms to

$$L(f) = -N \frac{f - f^{p-n}}{p} \tau(\tilde{\xi}^{1, \beta_1} + \frac{1}{\beta_1} (\ln t + 1))$$

(10)

Theorem 1

Let $(z-1)N(\beta, -1) - (p-n)(p + zN) p < 0$ and

$(p + zN) p (2N(\beta, -1) - p + n) - N(\beta, -1) (z-1) < 0$

where $z = k(p-2)+m$, $\beta$, mentioned number

Then for $t \to +\infty$ solution of task (1) - (2) bounded above by function $u_+(x, t)$

$$u(x,t) \leq u_+(x,t) = \tilde{f}(\tilde{\xi}(z), \tilde{f}(\tilde{\xi})) = (a - b\tilde{\xi}^{p-1})^{\frac{p-1}{m+k(p-2)+1}}$$

Proof

Show what $u_+(x, t)$ is the upper solution of task (1) - (2).

After putting $u_+(t, x)$ in (10) we get the following estimates

$$L(f) = -N \frac{f - f^{p-n}}{p} \tau(\tilde{\xi}^{1, \beta_1} + \frac{1}{\beta_1} (\ln t + 1))$$

To satisfy these estimates, the following conditions are sufficient

$$\frac{1}{\beta_1} + \frac{N}{p} - \frac{n}{\tau f^{\beta_1}} \geq 0$$

These conditions are true due to conditions of the theorem 2.

From the last two theorems it follows that for all large $t$ the self-similar solution $\tilde{u}(t) \tilde{f}(\tilde{\xi})$ is bounded above and below
III. RESULTS OF THE NUMERICAL EXPERIMENTS

Problem (1) - (2) has no analytical solution. Therefore, we will discuss result of the numerical experiments. To find a solution of problem at some point $x_i$, we are using numerical methods (see [6]-[8]). The resulting asymptotic of the solutions were used as an initial approximation for numerical computation.

A. 1-DIMENSIONAL CASE ($N=1$).

From problem (1) - (2) we have following 1-dimensional nonlinear parabolic equation with boundary and initial conditions

$$
\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial}{\partial x_i} \left[ \mathbf{u}^{m-1} \frac{\partial \mathbf{u}}{\partial x_i} \right] - u^\beta
$$

(11)

$$
u(0,x) = \phi_1(x) \geq 0, 0 \leq x \leq b$$

$$u(0,t) = \phi_2(t) > 0, t \in [0,T]$$

For problem (11) we construct the spatial grid $x_i$ with steps $h$

$$\Omega = \{x_i = ih, \ h > 0, \ i = 0,1,\ldots,n, \ hn = b\}$$

And temporal grid with $\tau$

$$\partial_x = \{t_j = j\tau, \ \tau > 0, \ j = 0,1,\ldots,m, \ \tau m = T\}$$

replace problem (11) implicit two-layer difference scheme and get the task of difference with fault $\Omega(h^2 + \tau^2)$

$$\left( y_{i+1} - y_i \right)_{\tau} = \frac{1}{h^2} \left[ K [a_{n+1} (y^{(n)})_{i+1}^m - y_i^m] - y_i^m \right] - \left( y_i^m \right), \ i = 1,2,\ldots,n-1; \ j = 0,1,\ldots,m-1$$

$$y_0 = u_0(x_i), \ i = 0,1,\ldots,n$$

$$y_n = \phi_2(t), \ j = 1,2,\ldots,m$$

(12)

Where $a_{i+1}$ and $a_i$ are nonlinear terms of the problem (11). In our case $a_{i+1}$ and $a_i$ are a thermal conductivity coefficients.

To calculate $a_{i+1}$ and $a_i$, use the following formulas [7]

$$a_{i+1}(y^{(i+1)}) = \frac{1}{2} \left[ \left( y_{i+1}^m \right)^m - \left( y_{i+1}^m \right)^{m-1} \frac{(y_{i+1}^{(i+1)})^m}{h} \right] +$$

$$+ \left( y_{i+1}^m \right)^m - \left( y_{i+1}^m \right)^{m-1} \frac{(y_{i+1}^{(i+1)})^m}{h} \right]$$

$$a_i(y^{(i)}) = \frac{1}{2} \left[ \left( y_{i}^m \right)^m - \left( y_{i}^m \right)^{m-1} \frac{(y_{i}^{(i)})^m}{h} \right] +$$

$$+ \left( y_{i}^m \right)^m - \left( y_{i}^m \right)^{m-1} \frac{(y_{i}^{(i)})^m}{h} \right]$$

(13)

when $i=1$ $a_i$ goes beyond the points, so following Milne formulas can be used[8]

$$\frac{du}{dx} \approx - y_i + 4 y_{i+1} - 3 y_{i+2}$$

$$\frac{du_i}{dx_i} \approx \left( \frac{y_i - y_{i-1}}{h} \right)$$

(14)

System of algebraic equation (12) is nonlinear is relative $y_{i+1}^m$.

For solve a system of nonlinear equations (12), we apply an iterative method and obtain following system of algebraic equation [6]

$$\frac{y_{i+1} - y_i}{\tau} = \frac{1}{h^2} \left[ K [a_{n+1} (y^{(i+1)}) (y_{i+1}^m - y_i^m)] - \left( y_{i+1}^m \right) \right] - \left( y_i^m \right)$$

(15)

where $s = 0,1,2,\ldots$ initial and boundary conditions unchanged

Now system of algebraic equation (15) is linear is relative $y_i^m$. As the initial iteration for $y_1^m$ $y$ getting from the previous step time $y_i^m$ is getting from the previous step time $0$. When counting by an iterative scheme, the accuracy $\varepsilon$ of the iteration is set and the process continues until execution the following conditions

$$\max_{0 \leq i \leq n} \left| y_i^m - y_i \right| < \varepsilon$$

Remark. In all numerical calculations we take $\varepsilon = 10^{-3}$

Let following notation

$$y_1^m = y_1, \ y_1^m = y_1$$

From different scheme (15) we will find tridiagonal matrix coefficients A, B, C, F and solve following system linear equations by method Thomas [6]

$$\begin{align*}
A_i y_{i-1}^m + C_i y_i^m + B_i y_{i+1}^m &= F_i, \ i = 1,2,\ldots,n-1, \\
y_0 &= \chi_1 y_1 + \mu_1, \\
y_n &= \chi_2 y_{n-1} + \mu_2
\end{align*}$$

where

$$\begin{align*}
A_i &= \tau a_i (a^{(i)}) |a_i|, \\
B_i &= \tau a_i (a^{(i)}) |a_i|, \\
C_i &= A_i + B_i + 1, \\
F_i &= y_i - y_i^m
\end{align*}$$

(16)

B. Visualization

Notice that very important to found appropriate beginning approximation of solution. Therefore, an initial and a boundary values are calculated using the founded above following asymptotic

$$u_0(t,x) = (t \ln t) \left( \frac{1}{\beta} \right), \beta = m + k(p-2) + \frac{s}{p}$$

$$\tilde{f}(x) = (a - b x^{p+1}) s^{p+1} \beta, s = pN / (p - n), p > n,$$

$$b = (m + k(p-2) - 1) (k^{p+1}) (p^{p+1}) m + k(p-2) - 1 > 0,$$
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\[ \xi = \varphi(x)[\tau(t)]^{-\frac{1}{\alpha_1}}, \varphi(x) = p(p-n)^{-1} x^{\frac{p-n}{p}} , n < p, \]

\[ \tau(t) = t^{\alpha_1} \ln^{\alpha_2} t, \]

\[ \alpha_1 = \frac{m+k(p-2)-1}{(p+(k(p-2)+m)s)p}, \]

\[ \alpha_2 = -\frac{m+k(p-2)-1}{(p+(k(p-2)+m)s)p} + 1 \]

Figure 1. p=3 k=2 m=2 n=2

Figure 2. p=3 k=2 m=2 n=1

Figure 3. p=3 k=2 m=5 n=2

Figure 4. p=4 k=2 m=2 n=2
All this suggest that parameter $\beta$ is very sensitive to parameters $k$, $p$, $m$. Besides at $m + k(p - 2) = 1$ the asymptotic to the solution will also change, which as in the critical case, can discover new effects, properties, estimates.

For the equation with variable density which is given below, the critical exponent of parameter $\beta$ is also found. With it helps search the asymptotic of the solution with critical parameter possible.

$$|x|^\gamma \frac{\partial u}{\partial t} = \nabla \left( u^{m-1} |\nabla u|^{p-2} \nabla u \right) - |x|^\gamma u^\beta,$$

$$u(x,0) = u_0(x), \ x \in \mathbb{R}^N$$

$$u(x,t) = w(\rho(x,t))$$

$$\frac{\partial w}{\partial t} = \phi^{\gamma-1} \frac{\partial}{\partial \phi} \left( \phi^{\gamma-1} w^{m-1} \left| \frac{\partial w}{\partial \phi} \right|^{p-2} \frac{\partial w}{\partial \phi} \right) - w^\beta$$

$$\phi(x) = p| x |^\gamma (p-1) / (p-l), \ s = p(N-l) / (p-l), \ N > 1, l < p$$

$$\beta_* = k(p-2) + p / s = k(p-2) + \frac{p-1}{N-l}$$

The search for new effects of a variable density problem for a critical value of parameter $\beta$ is a very important and interesting study. It expands the possibilities of applied application of this study and this equation a more generalized case of problem (1) - (2).

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