International Journal of Innovative Technology and Exploring Engineering (IJITEE)
ISSN: 2278-3075 (Online), Volume-9 Issue-1, November 2019

# Stability, Bifurcation, Chaos : Discrete Prey Predator Model with Step Size

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***Abstract: In this work titled Stability, Bifurcation, Chaos: Discrete prey predator model with step size, by Forward Euler Scheme method the discrete form is obtained. Equilibrium states are calculated and the stability of the equilibrium states and dynamical nature of the model are examined in the closed first quadrant $R_+^2$ with the help of variation matrix. It is observed that the system is sensitive to the initial conditions and also to parameter values. The dynamical nature of the model is investigated with the assistance of Lyapunov Exponent, bifurcation diagrams, phase portraits and chaotic behavior of the system is identified. Numerical simulations validate the theoretical observations.***

***Keywords : Population Dynamics, Discrete Time, Fixed points, Stability, Bifurcation theory, Lyapunov Exponent, Chaos.***

## I. INTRODUCTION

In the present age as life science research grows to be increasingly quantitative, the need for mathematical modeling becomes ever more significant. Mathematical modeling is a scientific method to depict in conceptual world the real world phenomena using mathematical language and tools. It enables to construct simplified representations of complex processes and phenomena. The benefit of mathematical models is that they can be investigated in a precise way by means of mathematical theory and properties.

Each system is formed with a set of parameter variables and an order of evolution of these parameters. By using these rules to the model's parameters, the qualitative and quantitative properties of the model can be analyzed and the results interpreted to make decisions, predict the future and provide hypotheses that can be verified [3].

Mathematical modeling draws engineers and scientists alike as it finds application in various fields, ranging from Anthropology to Artificial Intelligence, from Ecology to Chemical Engineering, from Military Applications to Industry, from Psychology to Space Science.

Revised Manuscript Received on November 30, 2019.
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## II. LOTKA - VOLTERRA DYNAMICAL SYSTEM

Among the species in the surroundings, there are different types of interactions such as predation, mutualism, competition and parasitism. Whenever an organism consumes another living organism, this interaction is termed as 'predation'. Due to its significance and universal existence, the study of prey predator models is of atmost interest as it helps to forecast and also analyze the behavior of prey and predator in the environment.The study of the dynamical nature of the interactions between prey - predator living in the same circumstances will continue to be one of the important topics of research in mathematical ecology.

Interaction of different biological species has been an interesting model,introduced by Volterra. The two species type of this model was discussed by Lotka. The Lotka-Volterra model has been considered as a basis for most ecological processes due its simplicity in nature and its ability to explain complex qualitative behavior. Several authors investigated the dynamical nature of the predator-prey model and contributed to the progress of the population dynamics over the years in the field of ecology [5], [6], [12], [13].

### A. Continuous Time Model

Prey-Predator interactions which is of great interest for ecologists, biologists and mathematicians alike are described by differential equations or dynamical systems (continuous and discrete). Researchers have studied the prey predator models and analyzed the dynamical behaviors. [4] Carmen ansidered and analyzed the following predator-prey model:

$$\begin{aligned} Dx_t &= x_t - x_t^2 - x_t y_t \\ Dy_t &= \beta x_t y_t - \alpha y_t \end{aligned} \qquad (1)$$

where $D = \frac{d}{dt}$ and $\alpha, \beta$ are positive constants.

### B. Discrete Time Model

In recent years, [1], [2], [8], [9] population dynamics of discrete-time have come to the fore front due to the following reasons: Firstly, they are more appropriate than continuous time systems to define populations with non overlapping generations. Secondly, they yield more rich and complex dynamical behavior than continuous-time systems. Finally, more accurate numerical simulations are obtained from discrete-time systems in comparision to continuous-time systems. Applying the forward Euler scheme to system (1), we get the modified discrete-time prey- predator model as follows:

*Retrieval Number: A4866119119/2019©BEIESP*
*DOI: 10.35940/ijitee.A4866.119119*
*Journal Website: www.ijitee.org*



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$$\begin{aligned} x_{t+1} &= x_t + h\left[x_t - x_t^2 - x_t y_t\right] \\ y_{t+1} &= y_t + h\left[\beta x_t y_t - \alpha y_t\right] \end{aligned} \quad (2)$$

where $x_t$ and $y_t$ denote the prey population density and the predator population density at time $t$, respectively, $h > 0$ is step size, $\beta > 0$ represents the coefficient of conversion and $\alpha > 0$ is the mortality rate of the predator.

### C. Fixed Points and Variation Matrix of System (2)

Now we find the fixed points of the system (2) by formulating the variation matrix of the model (2). To determine the fixed points we solve the algebraic system from equation (2) and obtain the following fixed points,

- $S_0 = (0,0)$ is trivial point.
- $S_1 = (1,0)$ is semi trivial point when the predator goes to extinction.
- $S_2 = \left(\frac{\alpha}{\beta}, 1-\frac{\alpha}{\beta}\right)$ is the interior fixed point which is feasible only if $\beta > \alpha$.

Now, we examine the nature of system (2) around the above fixed points. The stability behavior of system (2) can be discussed by computing the variation matrix for each of the fixed points. The variation matrix (3) is given by

$$V(x,y) = \begin{bmatrix} 1+h(1-2x-y) & -hx \\ \beta hy & 1-\alpha h+\beta hx \end{bmatrix}. \quad (3)$$

The characteristic equation (4) is

$$\Theta(\tau) = \tau^2 - T\tau + D = 0 \quad (4)$$

where $T$ and $D$ are the trace and determinant of the variation matrix $V(x,y)$, expressed as $T = 2 - h[\alpha + x(2-\beta) - 1 + y]$ and $D = h^2[(\beta x - a)(1-2x) + ay] + 1 + h[1-\alpha - y + x(\beta - 2)]$. We follow lemma [7], [10], [11], [14] to investigate the stability of the fixed point, which can be evaluated by the relations between the characteristic equation and roots of the equation.

**Lemma 1.** Let us consider equation (4) and $\tau_1, \tau_2$ be the roots of $\Theta(\tau) = 0$. Suppose that $\Theta(1) > 0$. Then we have

- $|\tau_1| < 1$ and $|\tau_2| < 1 \iff \Theta(-1) > 0$ and $\Theta(0) < 1$ (The fixed point $(x^*, y^*)$ is stable(sink)).
- $|\tau_1| < 1$ and $|\tau_2| > 1$ (or $|\tau_1| > 1$ and $|\tau_2| < 1$) $\iff \Theta(-1) < 0$. (The fixed point $(x^*, y^*)$ is saddle point).
- $|\tau_1| > 1$ and $|\tau_2| > 1 \iff \Theta(-1) > 0$ and $\Theta(0) > 1$. (The fixed point $(x^*, y^*)$ is unstable(source)).
- $|\tau_1| = -1$ and $|\tau_2| \neq 1 \iff \Theta(-1) = 0$ and $T \neq 0$ and $2$.
- $\tau_1$ and $\tau_2$ are complex and $|\tau_1| = |\tau_2| \iff T^2 - 4D < 0$ and $\Theta(0) = 1$.

## III. STABILITY ANALYSIS AND NUMERICAL SIMULATIONS

Asymptotic Stability means all the nearby initial values tend to the fixed point, on the other hand if all the nearby initial points approach the nearby fixed point then it is known as stability. System (2), holds the following propositions and corollaries.

**Proposition 1.**

The trivial fixed point $S_0$ is a

- sink if $\frac{1}{1+h} < \alpha < \frac{3+2h}{2h+h^2}$.
- source if $\alpha < \min\left\{\frac{3+2h}{2h+h^2}, \frac{1}{1+h}\right\}$.

**Proof.** The variation matrix at fixed point $S_0$ is as follows:

$$V(S_0) = \begin{bmatrix} 1+h & 0 \\ 0 & 1-\alpha h \end{bmatrix}.$$

The characteristic equation of $V(S_0)$ satisfies $\Theta(\tau) = \tau^2 - T\tau + D = 0$, where $T = h(1-\alpha) + 2$ and $D = (1-\alpha h)(1+h)$. Also the roots of the variation matrix $V(S_0)$ are $\tau_1 = 1+h$ & $\tau_2 = 1-\alpha h$, which may be less than or greater than one. So it could be either a sink or a source respectively.

**Corollary 1.** The point $S_0$ is saddle if $\alpha > \frac{3+2h}{2h+h^2}$ and Non-hyperbolic if either $\alpha = \frac{3+2h}{2h+h^2}$ or $\alpha = \frac{1}{1+h}$.

**Proposition 2.**

The semi trivial fixed point $S_1$ is a

- sink if $\alpha - \frac{2}{h} < \beta < \alpha + \frac{1}{1-h}$.
- source if $\beta > \max\left\{\alpha - \frac{2}{h}, \alpha + \frac{1}{1-h}\right\}$.

**Proof.** Using the semi trivial fixed point $S_1$ in the variation matrix we have

$$V(S_1) = \begin{bmatrix} 1-h & -h \\ 0 & 1+h(\beta-\alpha) \end{bmatrix}$$

Solving the matrix $V(S_1)$ we get the characteristic equation $\Theta(\tau) = \tau^2 - T\tau + D = 0$, where $T = 2 + h[\beta - \alpha - 1]$ and $D = (1-h)[1 + h(\beta - \alpha)]$. Also $\tau_1 = 1-h$ & $\tau_2 = 1 + h[\beta - \alpha]$ are eigen values of $V(S_1)$. Using the Lemma 1, the semi trivial fixed point is a sink or a source respectively. This completes the proof.

**Corollary 2.** If $\beta < \alpha - \frac{2}{h}$, then the semi trivial fixed point $S_1$ is saddle and $S_1$ is Non-hyperbolic if either $\beta = \alpha - \frac{2}{h}$ or $\beta = \alpha + \frac{1}{1-h}$.

We now analyze the stability of interior fixed point $S_2$ by finding the variation matrix

$$V(S_2) = \begin{bmatrix} 1-\frac{\alpha h}{\beta} & -\frac{\alpha h}{\beta} \\ h(\beta-\alpha) & 1 \end{bmatrix}.$$

The characteristic equation of $V(S_2)$ is given by $\Theta(\tau) = \tau^2 - T\tau + D = 0$, where $T = 2 - \frac{\alpha h}{\beta}$ and $D = 1 - \frac{\alpha h}{\beta} + (\beta - \alpha)\frac{h^2\alpha}{\beta}$.

Then simple calculations, yield

the eigen values $\tau_1, \tau_2$. Considering Lemma 1, we have

(A₁.) $\frac{\alpha h[2+\alpha h]}{4+\alpha h^2} < \beta < \alpha + \frac{1}{h}$ and $|\tau_{1,2}| < 1$.

(A₂.) $\beta > \max\left\{\frac{\alpha h[2+\alpha h]}{4+\alpha h^2}, \alpha + \frac{1}{h}\right\}$ and $|\tau_{1,2}| > 1$.

(A₃.) $\beta < \frac{\alpha h[2+\alpha h]}{4+\alpha h^2}$ & $|\tau_1| < 1, |\tau_2| > 1$ (or $|\tau_1| > 1, |\tau_2| < 1$).

(A₄.) $\beta = \frac{\alpha h[2+\alpha h]}{4+\alpha h^2}$ or $\beta = \alpha + \frac{1}{h}$ and $|\tau_{1,2}| = 1$.

**Proposition 3.**
If $\beta > \alpha$, then following results are true:

(1.) If $(A_1)$ holds, then $S_2$ is stable (sink).

(2.) If $(A_2)$ holds, then $S_2$ is unstable (source).

(3.) If $(A_3)$ holds, then $S_2$ is saddle.

(4.) If $(A_4)$ holds, then $S_2$ is non-hyperbolic.

Now, we present the simulations and numerical evidence to understand the dynamical nature of system (2). For example, consider the parameter values $h = 0.6; \alpha = 0.9; \beta = 2.5$, the interior steady state of the system (2) is $S_2 = (0.36, 0.64)$ and the variation matrix is $V(0.36, 0.64) = \begin{bmatrix} 0.7840 & -0.2160 \\ 0.9600 & 1 \end{bmatrix}$. Here $T = 1.7840$, $D = 0.9914$ and $\Theta(-1) = 3.7754 > 1$, $\Theta(0) = 0.9914 < 1$. Also the eigen values of $S_2$ are $|\tau_{1,2}| = 0.9957 < 1$. Using the Proposition 3, we see that the system is stable (see Fig-1).

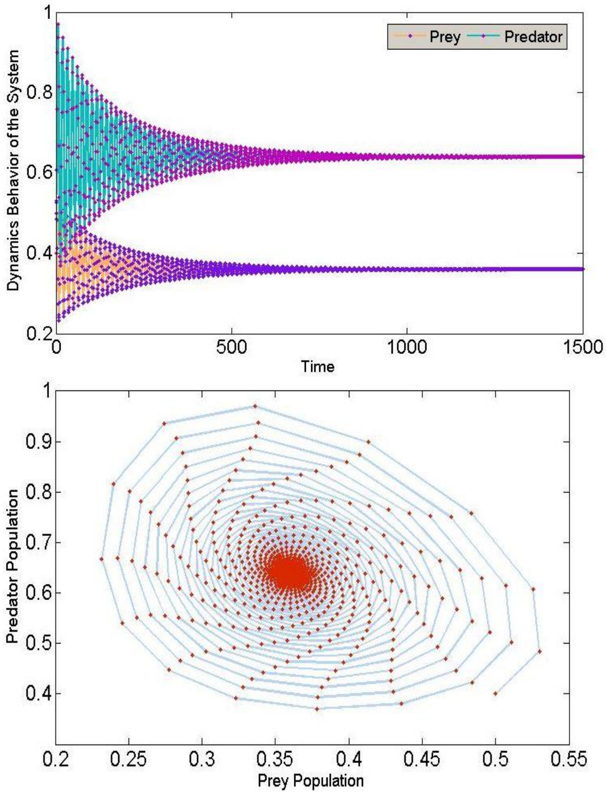


**Fig. 1. Stability of Interior Steady State of the System (2)**

Let us consider another example with the values $h = 1.45; \alpha = 1.6; \beta = 2.3$, assuming the initial point $x_0 = 0.5, y_0 = 0.4$ and the variation matrix of system (2) is $V(S_2) = \begin{bmatrix} -0.0087 & -1.0087 \\ 1.0150 & 1 \end{bmatrix}$. Here $T = 0.9913, D = 1.0151$ and $\Theta(-1) = 3.0064 > 1$ $\Theta(0) = 1.0151 > 1$. Also $|\tau_{1,2}| = 1.0075 > 1$ are the eigen values of $S_2$. It is seen that the system is unstable by applying the Proposition 3. (see Fig. 2).

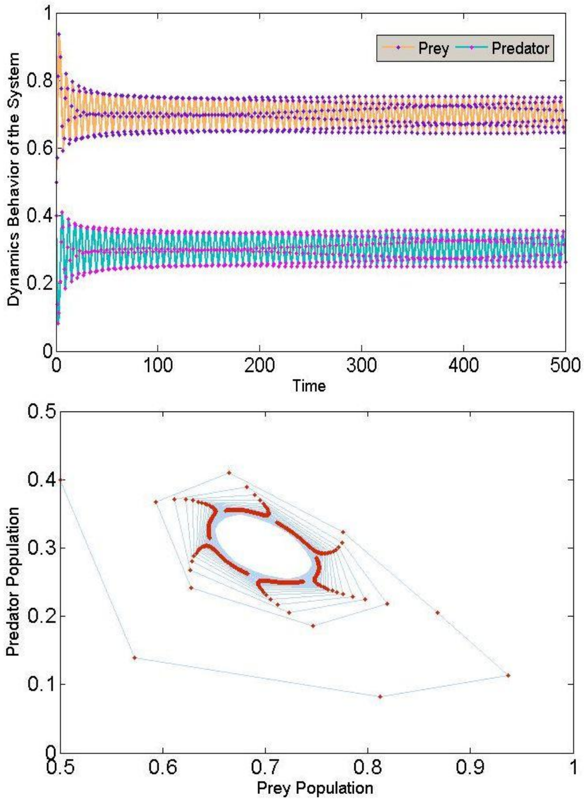


**Fig. 2. Unstable of Interior Steady State of the System (2)**

## IV. EXISTENCE OF BIFURCATION

In this section, bifurcation theory is applied to find the existence criteria for Neimark-Sacker bifurcation at the co-existence fixed point $S_2$ of system (2). For this, we fix $h$ as bifurcation parameter and choosing $h \equiv h_0 = \frac{1}{\beta - \alpha}$. At $h = h_0$, the characteristic equation is

$$\Omega(\rho) = \rho^2 - \left[2 + \frac{\alpha}{\beta(\alpha-\beta)}\right]\rho + 1.$$

It can be presented as follows

$$\Omega(\rho) = \left[\rho - \left(\frac{\alpha + 2\beta(\alpha-\beta)}{2\beta(\alpha-\beta)} \pm \frac{1}{2\beta(\alpha-\beta)}\sqrt{\alpha^2 + 4\alpha\beta(\alpha-\beta)}\right)\right] \quad (5)$$

Then, the roots of (5) are

$$\rho_{1,2} = \frac{\alpha + 2\beta(\alpha-\beta)}{2\beta(\alpha-\beta)} \pm \frac{1}{2\beta(\alpha-\beta)}\sqrt{\alpha^2 + 4\alpha\beta(\alpha-\beta)}.$$

Moreover, consider the following set,

$$S_{NSB} = \left\{(h_0, \alpha, \beta) : \alpha > 0, \beta > 0, h_0 = \frac{1}{\beta - \alpha}, |\rho_{1,2}| = 1\right\} \quad (6)$$

We illustrate the bifurcation diagrams to establish the analytic results of (6) and the nature of system (2) as the parameters vary.

Bifurcation diagrams of system (2) in $(h-x)$ and $(h-y)$ planes displays the dynamical behavior of the system as the parameter $\alpha = 1.6, \beta = 2.3$ are fixed and $h$ is varied. From Fig.3, it is clear that the curve with initial conditions $(0.5, 0.4)$ comes close to the co-existence fixed point $S_2$ which is stable if $h < 1.4286$, and a periodic doubling occurs at $h = 1.4286$.

As $h$ increases, the positive steady state turns out to be unstable through a periodic doubling bifurcation and the appearance of chaos is visible in the simulation. Also applying the above parameter values to the characteristic equation (5), we have $\tau_{1,2} = 0.5031 \pm i0.8642$ and $|\tau_{1,2}| = 1$, which fulfills the criteria for Neimark-Sacker bifurcation.

To study the chaotic behaviour of the model we compute Maximal Lyapunov Exponent depending on $h$ and also undertake sensitivity analysis of the system to initial conditions.The presence of positive Lyapunov Exponent is characteristic of chaos. Moreover by comparing the standard bifurcation diagram, one obtains a better understanding of the particular properties of the dynamical behavior of system (2). Fig.3 exhibits the associated Maximal Lyapunov Exponent of system (2) as function in $h$. From Fig.3, the degree of the local stability for $h \in (1.3, 2.1)$ is established. For $h > 1.4286$ Maximal Lyapunov Exponent is positive confirming the existence of chaos.

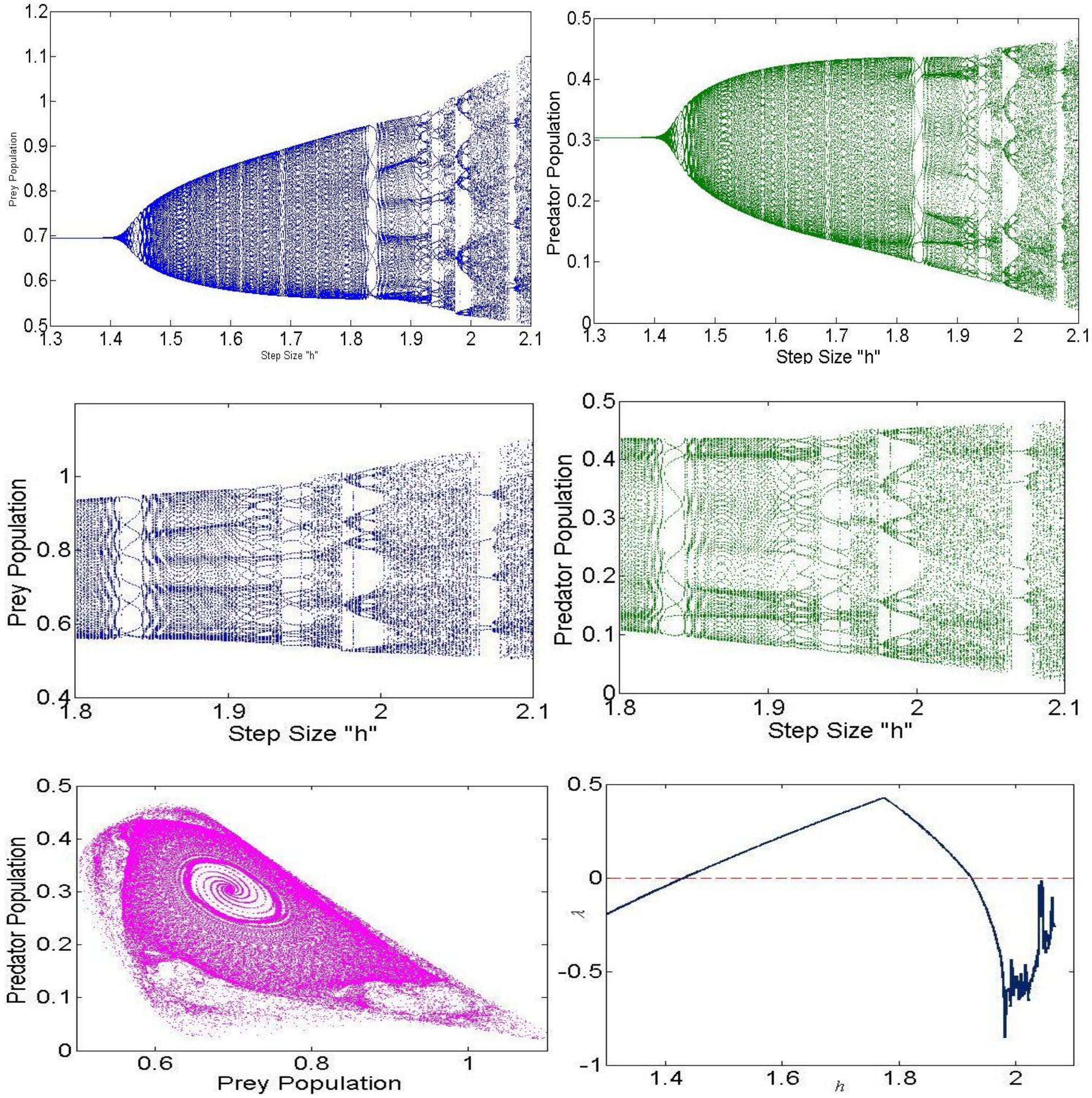


**Fig. 3. Bifurcation Structure of the System (2)**

The phase space for a range of values of $h$ related to Fig. 3 are displayed in Fig. 4, which clearly captures the bifurcation as the parameter $h$ varies. As the value of $h$ increases, we observe the transition from stability to instability and exhibits chaos.

*Retrieval Number: A4866119119/2019©BEIESP*
*DOI: 10.35940/ijitee.A4866.119119*
*Journal Website: www.ijitee.org*



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At $h = 1.3$, the phase portraits begin with stability and at $h = 1.5$ smooth invariant circle appears and for later values the circle is stretched and shrunk. As $h$ increases, the circles disappear and there are some cascades of periodic doubling from $h = 1.83$ to $h = 2.07$ leading to attracting chaotic sets at $h = 2.1$.

### A. Sensitive Analysis on Initial Condition

A characteristic of chaos is the sensitivity to initial conditions. In order to test the sensitivity of the system (2) to initial values, two paths are calculated with initial conditions $(x_0, y_0)$ and $(x_0 + 0.0001, y_0)$ as well as $(x_0, y_0)$ and $(x_0, y_0 + 0.0001)$ respectively. The established result is displayed in Fig.-5 & 6. Initially, paths are impossible to differentiate; however there is a difference between them after a number of iterations and this diffference builds up rapidly.

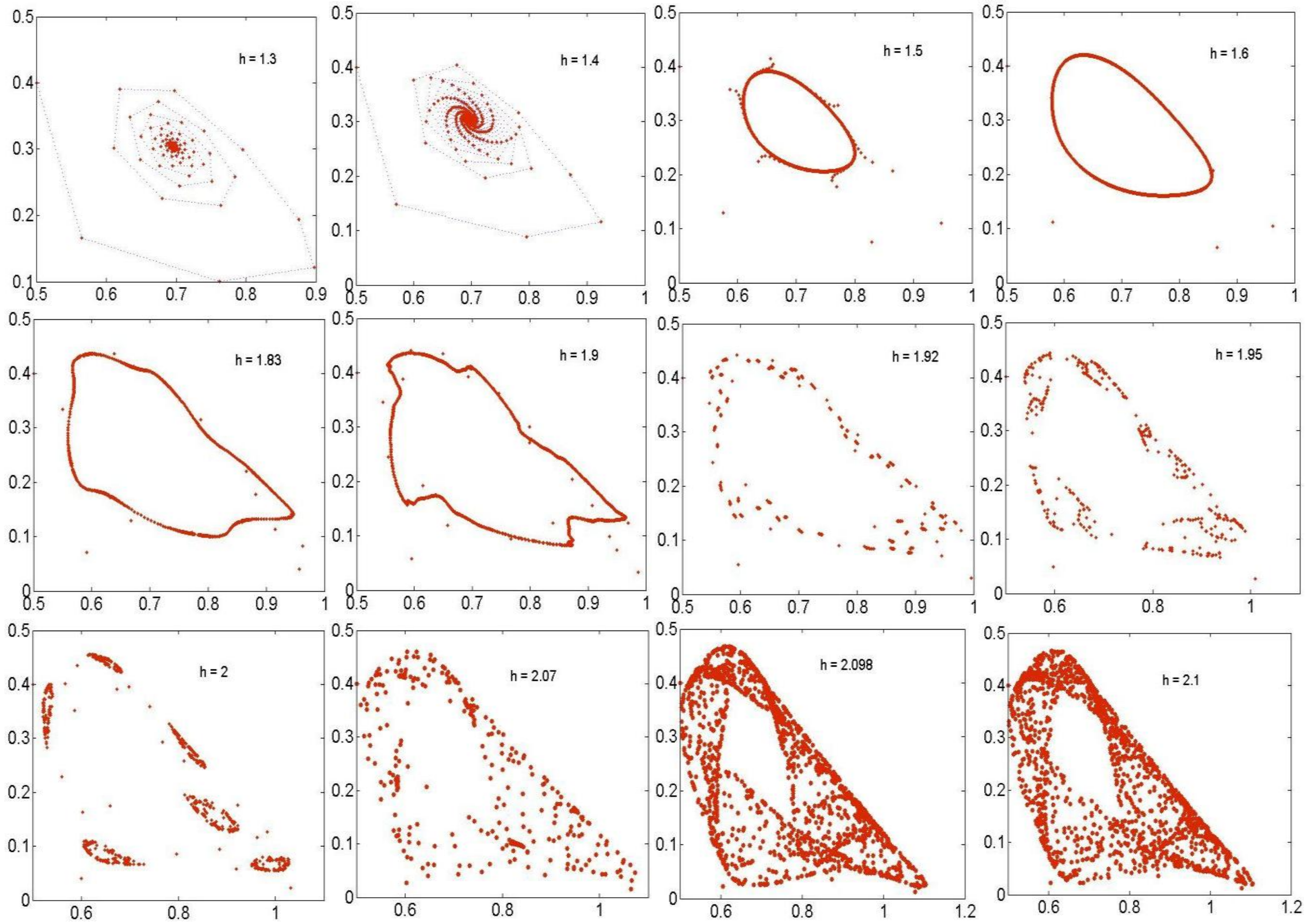


**Fig. 4. Various Phase Portraits of the System (2).**

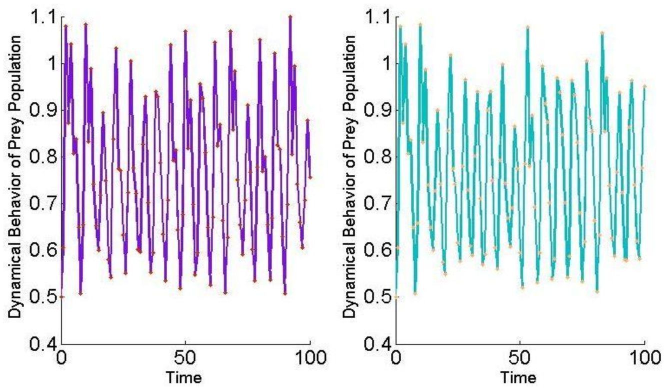


**Fig. 5. Sensitive Analysis for the Prey Population of System (2)**

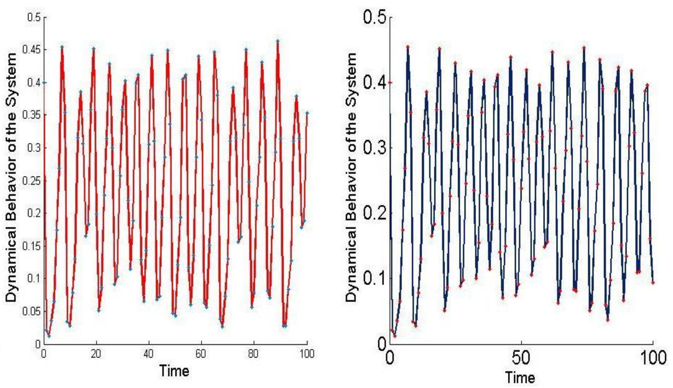


**Fig. 6. Sensitive Analysis for the Predator Population of system (2)**

## V. CONCLUSION

A two dimensional modified discrete prey-predator system is discussed in this paper and it is seen that the system exhibits rich dynamics. Foremost the existential conditions for the fixed points of the system are derived and also the conditions necessary to analyze the stability of the model. It is revealed that the system goes through Neimark-Sacker bifurcation and also essential criteria for the existence of Neimark - Sacker Bifurcation is obtained. For the co-existence fixed point the system is stable if the step size parameter $h$ lies in the range of $1.3 < h < 1.4286$, it is unstable for $h > 1.4286$ and at $h = 1.4286$ Neimark - Sacker Bifurcation occurs.

It is interesting to note that system displays rich dynamical nature such as periodic -2 orbits, invariant cycles, cascade of periodic doubling leading to chaotic sets.

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