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Return Time Probability Distribution of a Finite Markov Chain

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Abstract*: In this paper we have considered a finite discrete Markov chain and derived a recurrence relation for the calculating the return time probability distribution. The mean recurrence time is also calculated. Return time distribution helps to identify the most frequently visited states. Return time distribution plays a vital role in the classification of Markov chain. These concepts are illustrated through an example.

Keywords* : Discrete Markov Chain, Ergodic Markov Chain , Limiting distribution, Mean recurrence time, Return time probability, Transition probability matrix.

I. INTRODUCTION

Any real time problem can be solved efficiently using mathematical models. Markov model is one such that tool. A Markov model is a collection of random variables such that the future behaviour depends only on the present and not on the past.[1,2] This is applied in queuing theory, process engineering, reliability, etc. The problem of forecasting bike station transfer demand addressed using Markov chain[3]. In [4], conditions under which the sum of random variables in Markov chain converges to Poisson type distribution. Many researchers have studied the properties and limiting distribution of the underlying system using Markov models [5,6,7].

In discrete Markov chains, it is significant to calculate the probability of observing m visits to some state in n trails. In this paper, we have considered a finite Markov chain and calculated the return time probability distribution. The mean recurrence time is also computed. This can be used to identify the frequently visited states. The mean recurrence time for such states can also be calculated. The website of any organisation can be modelled as a Markov chain, the states being the web pages and the transitions between the web pages representing the transition probabilities. Calculation of return time distribution helps to identify the frequently accessible web page. This can be used to estimate the reliability of the website [8,9].

In section II, we have given the necessary concepts. A recursive relation for calculating the return time distribution of a finite Markov chain is presented in section III, illustration in section IV and finally concluded.

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II. PRELIMINARIES

Consider a generic Markov chain in discrete time on some space X. For such a process all transition probabilities can be described as products of the one period transition probabilities kernel $P(x, A)$ [1,4,5].

Here, $x \in X$ is the initial condition and $A \in B(X)$ takes place on a set of subsets of the state space.

$$P(x_{t+1} \in A / x_t = x) = P(x, A)$$

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$$P(x_{t+2} \in A / x_t = x) = \int_X P(x, dy)\, P(y, A) \quad (1)$$

$$P(x_{t+k} \in A / x_t = x) = \int_X P(x, dy)\, P^{k-1}(y, A)$$

Given the initial condition time is irrelevant and the t-subsripts disappear on the RHS of (1). If there is some set α such that $P(x_{t+k} \in A / x_t \in \alpha)$ is equal of all $x \in \alpha$, then return to set α will be regeneration times of the chain. For a discrete Markov chain any state $i \in X$ will have this property. In general such a set is known as an atom of the Markov chain.

Define the occupation time on a finite sample η_A as $\eta_A = \sum_{t=1}^{n} I_{x_t \in A}$ where I_x is the indicator function of the event x. We seek probability distribution $\phi(m, n) = P(\eta_A = m)$ for observations on some given sample $\{x_t\}_{t=1}^{n}$.

Let A be the set of interest and presume that visits to set A define regeneration times of the chain. Consider the probability that there is only one visit to A in the sample $\{x_t\}$. This is the probability that the first return occurs in less than or equal to n transitions and that the return occurs in less than or equal to n transitions and the second return takes place after *n* transitions. This is $\phi(1, n) . \phi(2, n)$ represents the probability of having two returns to A is less than or equal to n transitions and the third return takes place after *n* transitions.

We derive an recursive expression for $\phi(m, n)$ for a finite state Markov chain in the following section.

III.RETURN TIME DISTRIBUTION FOR FINITE MARKOV CHAIN

In this section we derive a recursive expression for return calculating the return time probabilities$\emptyset(m, n)$,$n > 1, m < n$ of a finite Markov chain.

Consider an ergodic Markov chain with three states 0, 1, 2 with probability transition matrix

$$P = \begin{pmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{pmatrix}.$$

We calculate the return time probabilities for state 0.

Consider the states 1 and 2 as $\acute{0}$.

Now, $p_{0\acute{0}} = p_{01} + p_{02} = 1 - p_{00}$;

$$p_{0\acute{0}}{}^{n} = (p_{01} + p_{02})^{n} = \sum_{i=0}^{n} n_{C_i} p_{01}{}^{i} p_{02}{}^{n-i}$$

$$= (1 - p_{00})^{n} = \sum_{i=0}^{n} n_{C_i} (-p_{00})^{i}$$

$p_{\acute{0}0} = p_{10} + p_{20}$;

$$p_{\acute{0}0}{}^{n} = \sum_{i=0}^{n} n_{C_i} p_{10}{}^{i} p_{20}{}^{n-i}$$

$p_{\acute{0}\acute{0}} = (1 - p_{10})(1 - p_{20})$;

$$p_{\acute{0}\acute{0}}{}^{n} = (1 - p_{10})^{n}(1 - p_{20})^{n}$$

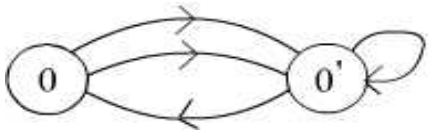
$$= \left(\sum_{i=0}^{n} n_{C_i} (-p_{10})^{i}\right)\left(\sum_{j=0}^{n} n_{C_j} (-p_{20})^{j}\right)$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n} n_{C_i} n_{C_j} (-p_{10})^{i} (-p_{20})^{j}$$

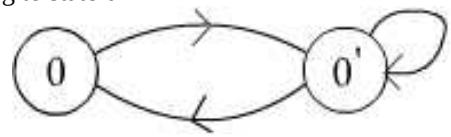
Consider $\emptyset(1, n)$, representing the probability of one return to state 0 in n transitions. This can happen in the following mutually exclusive ways:

(i) Return to state 0 in the first transition, moving to state $\acute{0}$ and having n-2 transitionsat $\acute{0}$.

(ii) Moving to state $\acute{0}$ and making one return to state 0 out of n-2 transitions



(iii) Moving to $\acute{0}$, having n-2 transitions and then returning to state 0

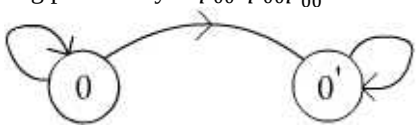


Hence,

$\emptyset(1, n) = p_{00} p_{0\acute{0}} p_{\acute{0}\acute{0}}{}^{n-2} + (n-2) p_{0\acute{0}}{}^{2} p_{\acute{0}0}\, p_{\acute{0}\acute{0}}{}^{n-3} + p_{0\acute{0}} p_{\acute{0}0} p_{\acute{0}\acute{0}}{}^{n-2}$.

Next, consider,$\emptyset(2, n)$, probability of having two returns to state 0 in n transitions. Starting at state 0 the nth transition can end at state $\acute{0}$ or at state 0. The following are the various ways in which this event can happen along with the respective probabilities:

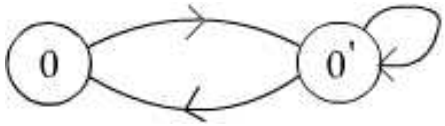
(i) Returning to state 0 in the first two transitions, moving to state $\acute{0}$ and having n-3 transitions at $\acute{0}$. The corresponding probability is $p_{00}{}^{2} p_{0\acute{0}} p_{\acute{0}\acute{0}}{}^{n-3}$.



(ii) Returning to state 0 in the first transition, moving to state $\acute{0}$ and in the remaining n-2 transitions having exactly one return to state 0. The associated probability is

$p_{00} p_{0\acute{0}} \emptyset(1, n-2)$.

(iii)Moving to state $\acute{0}$, first return to state 0 in the second transition and out of the remaining n-2 transitions one return to state 0. The corresponding probability is $p_{0\acute{0}} p_{\acute{0}0} \emptyset(1, n-2)$.



Hence, $\emptyset(2, n) = p_{00}{}^{2} p_{0\acute{0}} p_{\acute{0}\acute{0}}{}^{n-3} + p_{00} p_{0\acute{0}} \emptyset(1, n-2) + p_{0\acute{0}} p_{\acute{0}0} \emptyset(1, n-2)$.

Similarly, $\emptyset(3, n) = p_{00}{}^{3} p_{0\acute{0}} p_{\acute{0}\acute{0}}{}^{n-4} + p_{00}{}^{2} \emptyset(1, n-2) + p_{00} \emptyset(2, n-1) + p_{0\acute{0}} p_{\acute{0}0} \emptyset(2, n-2)$.

Proceeding in a similar manner, the recursive relation for $\emptyset(m, n)$ is

$$\emptyset(m, n) = p_{00}{}^{m} p_{0\acute{0}} p_{\acute{0}\acute{0}}{}^{n-(m+1)} + \sum_{j=1}^{m-1} p_{00}{}^{j} \emptyset(m-j, n-j) + p_{0\acute{0}} p_{\acute{0}0} \emptyset(m-1, n-2)$$

Also, note that, $\emptyset(n, n) = p_{00}{}^{n}$. Hence, the above recursive relation holds for $n > 1, m < n$.

This recursive relation itself can be used for a finite Markov chain with k states. Except 0 the remaining k-1 states are considered as $\acute{0}$. In this case,

$p_{0\acute{0}} = 1 - p_{00}$;

$p_{\acute{0}0} = p_{10} + p_{20} + \cdots + p_{k-10}$;

$p_{\acute{0}\acute{0}} = (1 - p_{10})(1 - p_{20}) \ldots (1 - p_{k-10})$.

For a two state Markov chain, this recursive expression coincides with []. Since $\emptyset(m, n)$is recursive, it is easier for computation. Using this the mean recurrence time in n transitions can be calculated as $f_n = \sum_{m \le n} m\, \emptyset(m, n)$.

IV. ILLUSTRATION

Consider a three state Markov chain with state space 0,1 and 2 having the transition probability matrix $P = \begin{pmatrix} 0.33 & 0.67 & 0 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.25 & 0.25 \end{pmatrix}$. We calculate the return time probabilities for all the states in four transitions. The mean recurrence time is also calculated. The results are tabulated in Table 1.

TABLE I
Return Time Distribution

States	0	1	2
$\phi(1,4)$	0.9877	0.5156	0.2899
$\phi(2,4)$	0.0934	0	0.0275
$\phi(3,4)$	0.2195	0	0.0773
$\phi(4,4)$	0.0123	0	0.0039
f_4	1.8453	0.5156	0.5924

From the table 1, based on the values of mean recurrence times we can conclude that state 0 is frequently visited than the other two states.

V. CONCLUSION

In discrete Markov chains it is significant to calculate the probability of observing m visits to some state in n trails. In this paper, a recursive expression to calculate the return time probabilities is presented. This expression can be used for any finite Markov chain. Return time distribution is essential to understand the dynamics of a Markov chain and understand the long run behavior of the system.

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