International Journal of Innovative Technology and Exploring Engineering (IJITEE)
ISSN: 2278-3075 (Online), Volume-9 Issue-1, November 2019

Optimal Search Parameters for A Random Pulsed-Point Source with the Required Accuracy

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Abstract*: The questions of creating high-speed algorithms for detecting and localizing point sources having a random distribution and manifesting themselves by generating instantaneous delta pulses at random times are described. The search is carried out by a system including one or more receiving devices, and is performed taking into account the requirements for localization accuracy. It is assumed that all receivers have freely tunable viewing windows. The optimal procedure is one that minimizes (in a statistical sense) the average localization time. It is established that even with relatively low requirements for localization accuracy, the optimal procedure consists of several stages (each such stage ends at the moment of the next pulse registration). In this case, it is possible receiving system to miss some pulses generated by the source during the optimal search. In the work, the optimal search parameters are calculated depending on the number of receiving devices and the required localization accuracy. The possibility of using the results in a multidimensional case is shown.

***Keywords*: search, optimal algorithms, pulsed-point source, localization.**

I. INTRODUCTION

In the thematic plan, the tasks and algorithms for the optimal search for random pulsed-point objects discussed in this paper arise in many scientific and technical applications. In digital recording and subsequent software processing of fast flowing dynamic processes of various physical nature, one of the most time-consuming and algorithmically complex moments is associated with the elimination of impulse noise created by point sources with a random spatial distribution. The successful solution of such problems requires a high-precision determination of the coordinates of radiation source objects, and in most practically important applications this needs to be done in a minimum (statistically) time.

Revised Manuscript Received on November 30, 2019.
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In astrophysics and cosmology [1-2], such problems are encountered when searching for bursters - flashing galactic X-ray sources. In the problems of technical diagnostics [3], such studies are required when developing methods for eliminating malfunctions that manifest themselves in the form of alternating failures. In modern areas of computer science, these methods are in demand in constructing algorithms for detecting small-sized objects in digital images [4], and, for example, in signal theory, these methods are used to evaluate the reliability of registration of random point fields [5]. Mathematically equivalent questions appear in the problems of detecting, localizing and tracking radiation source targets [6]. In most applied problems related to the localization of pulsed-point objects, the search is expected to be done in two- and three-dimensional areas. This paper is structured in such a way that its main content is devoted to the construction of time-optimal algorithms for one-dimensional search, and in the final section it is shown how the results obtained can be extended to the multidimensional case, and promising directions for further research are outlined.

II. TIME-OPTIMAL LOCALIZATION ALGORITHMS FOR SINGLE-RECEIVER SYSTEMS

Below, a pulsed-point random source will be understood as an object of negligible angular size (mathematical point) that has a probability density $f(x)$ within the search segment and generates infinitely short random pulses (delta functions) with Poisson intensity λ. The search for the object is carried out using a recording device (receiver) with a freely tunable viewing window in time. The pulse is detected if the point source at the moment of pulse's generation is in the viewing window of the receiving device. Otherwise, the impulse is considered as missed. When registering the next pulse, the position of the source on the coordinate axis is refined, as a result search segment narrows and the localization procedure is repeated until the next pulse is fixed, etc. It is necessary to find the source in minimum time, and to provide the required localization accuracy ε.

We are at first describe the ideas underlying the algorithms for searching for pulsed-point sources using a single receiving device.

Let the binary function $u(x,t) = 1$, if the point x at fixed time moment is in the receiver's viewing window, and $u(x,t) = 0$ otherwise, that describes the viewing window of the receiving device at time t. We obtain the ratio for the average time from the start of the search to the registration of the first pulse:

$$\langle\tau\rangle = \lambda\int_0^\infty dt\int_0^\infty dx\left[tf(x)u(x,t)\exp(-\lambda\int_0^t u(x,\xi)d\xi)\right]. \quad (1)$$

The problem of minimizing relation (1) comes down to finding the function *φ(x)* that minimizes the average search time

$$\langle\tau\rangle = \frac{1}{\lambda}\int\frac{f(x)}{\varphi(x)}dx \quad (2)$$

under conditions $\int\varphi(x)dx = \varepsilon,\ \ 0\le\varphi(x)\le 1.$

In the general case, the construction of an optimal one-stage search algorithm is associated with finding the function *φ(x, t)* – the relative load on the point *x* at time *t*, which minimizes the average localization time.

$$\langle\tau\rangle = \int dt\int dx f(x)\exp(-\lambda\int_0^t\varphi(x,\xi)d\xi)$$

provided that $0\le\varphi(x,t)\le 1,$ (3)

and for any *t*

$$\int\varphi(x,t)dx = \varepsilon. \quad (4)$$

To simplify further calculations, we introduce a function corresponding to the total time of point *x* staying in the viewing window for the entire period from the beginning of the search to the time *t*. Taking into account constraints (3) and (4), we introduce the indefinite Lagrange multiplier *μ(t)* [7]. Then the task of constructing an optimal search strategy is reduced to finding a function *α(x, t)* that minimizes the functional

$$\int dt\int dx\left[\exp(-\lambda\alpha(x,t))f(x)+\mu(t)\alpha(x,t)\right]$$

provided that

$$\int_{-\infty}^{\infty}\alpha(x,t)dx = \varepsilon t, \quad (5)$$

$$0\le\alpha(x,t)\le t.$$

The solution to this variational problem is the function

$$\alpha(x,t) = \begin{cases} 0, \frac{1}{\lambda}\ln\frac{\lambda f(x)}{\mu(t)} < 0; \\ \frac{1}{\lambda}\ln\frac{\lambda f(x)}{\mu(t)}, 0\le\frac{1}{\lambda}\ln\frac{\lambda f(x)}{\mu(t)}\le t; \\ t, t<\frac{1}{\lambda}\ln\frac{\lambda f(x)}{\mu(t)}, \end{cases} \quad (6)$$

where any binary function satisfying the conditions can be taken as the optimal search strategy *u(x, t)*

$$\int_0^t u(x,\xi)d\xi = \alpha(x,t); \qquad \int u(x,t)dx = \varepsilon.$$

In practice, the use of optimal search algorithms, as a rule, requires their rework and adaptation to the conditions of the problem being solved. The fact is that in cases where prior probability density of random source differs from uniform, most optimal algorithms cannot be physically implemented by moving a simply connected scanning window. Therefore, when developing real search engines, they usually resort to various simplifications, and exact analytical relationships are needed in order to correlate the output parameters of such systems with theoretically achievable limit values. It should be noted that in the second of the above algorithms, by default, it is assumed that the source intensity *λ* is known in advance.

If we do not limit ourselves to single-stage procedures, but consider the search algorithm as a multi-stage process (which ends after the registration of the *m*-th pulse), then the optimal strategy should deliver a minimum of functionality

$$\langle\tau\rangle = \sum_{k=1}^{m}\lambda^k\int_0^\infty dx f(x)\int\ldots\int t_k\times$$
$$\left\{\prod_{l=1}^{k}\left[dt_l u_l(x,\sum_{s=1}^{l}t_s/t_1,\ldots,t_{l-1})\exp(-\lambda\int_{\sum_{s=1}^{l-1}t_s}^{\sum_{s=1}^{l}t_s}u_l(x,\xi/t_1,\ldots t_{l-1})d\xi)\right]\right\} \quad (7)$$

under condition

$$\int u_m(x,t/t_1,\ldots,t_{m-1})dx = \varepsilon.$$

Here is the search strategy at the *i*-th step, provided that the fixed time intervals between the first *(i-1)* pulses were $t_1, t_2, \ldots, t_{i-1}$, respectively. In the general case, it seems unlikely to find the optimal strategy that delivers the minimum to functional (7) whether possible. At the same time, for an important special case, when *f(x)* = const, it is possible to obtain an analytical solution. Let be

$$f(x) = \begin{cases} \frac{1}{L}, & x\in[0,L], \\ 0, & x\notin[0,L], \end{cases}$$

so, there is no a priori information about the possible location of the source inside the search segment [*0, L*]. Obviously, in this case, at the first stage, the search efforts should be equally distributed between all the points of the search segment. Such a load can be achieved, for example, by scanning the segment [*0, L*] with an aperture of width W_1 (we neglect edge effects). When a pulse is detected, the search continues inside the aperture window of width W_1 using another aperture having a width of W_2. If *m*-stage search is considered, then scanning at the last stage should be performed by an aperture having a width *ε* (this is dictated by the conditions of the problem). With this organization of the search procedure, the average localization time of the pulse source will be

$$\langle\tau\rangle = \frac{1}{\lambda}\left(\frac{L}{W_1}+\frac{W_1}{W_2}+\frac{W_2}{W_3}+\ldots+\frac{W_{m-1}}{\varepsilon}\right). \quad (8)$$

If the value of *m* is fixed, then the minimum of expression (8) is achieved when

$$L/W_1 = W_1/W_2 = \ldots = W_{m-1}/\varepsilon = (L/\varepsilon)^{1/m}.$$

To verify this, it is enough to consistently equate to zero all partial derivatives of expression (8) with respect to variables $W_1, W_2, \ldots, W_{m-1}$:

$$\frac{1}{\lambda}\left(-\frac{L}{W_1^2}+\frac{1}{W_2}\right) = 0, \quad \text{(this leads to equality } \frac{L}{W_1}=\frac{W_1}{W_2}\text{);}$$

$$\frac{1}{\lambda}\left(-\frac{W_1}{W_2^2}+\frac{1}{W_3}\right) = 0, \quad \text{(this leads to equality } \frac{W_1}{W_2}=\frac{W_2}{W_3}\text{) ...}$$

$$\frac{1}{\lambda}\left(-\frac{W_i}{W_{i+1}^2}+\frac{1}{W_{i+2}}\right)=0, \quad \text{(this leads to equality } \frac{W_i}{W_{i+1}}=\frac{W_{i+1}}{W_{i+2}}\text{) ...}$$

$$\frac{1}{\lambda}\left(-\frac{W_{m-2}}{W_{m-1}^2}+\frac{1}{\varepsilon}\right)=0, \quad \text{(this leads to equality } \frac{W_{m-2}}{W_{m-1}}=\frac{W_{m-1}}{\varepsilon_1}\text{).}$$

If we additionally take into account that $\frac{L}{W_1}\times\frac{W_1}{W_2}\times\frac{W_2}{W_3}\times\ldots\times\frac{W_{m-1}}{\varepsilon}=\frac{L}{\varepsilon}$, then the average time of the optimal m-stage search (8) is converted into

$$\langle\tau_m\rangle_{opt}=(m/\lambda)(L/\varepsilon)^{1/m}. \qquad (9)$$

Here is the search strategy at the *i*-th step, provided that the fixed time intervals between the first (*i*-1) pulses were $t_1, t_2, \ldots, t_{i-1}$ respectively.

Now, using expression (9), we can find the optimal number of stages m_{opt} at which the average search time (8) reaches a global minimum. Since the function $xa^{1/x}$ for $a>1$ has only one minimum at $x=\ln(a)$, the optimal value of m_{opt} is always either *entier(ln(L/ε))* or *entier(ln(L ε))+1* (here *entier (z)* is the integer part of *z*), so we get that the following asymptotic relations are true:

$$L/W_1=W_1/W_2=\ldots=W_{m-1}/\varepsilon=e.$$

Thus, we have accurately calculated the parameters of the optimal search for a uniformly distributed random pulse source depending on the required localization accuracy, which are presented in the resulting tables.

Table-1. Required localization accuracy 1/4 ≤ (ε/L)<1

optimal number of stages m_{opt}	1
viewing windows of the receiving system at each of the m_{opt} stages of optimal search	$W_1=\varepsilon$
average localization time	$\frac{1}{\lambda}(\varepsilon/L)^{-1}$

Table-2. Required localization accuracy (2/3)6 ≤(ε/L)<1/4

optimal number of stages m_{opt}	2
viewing windows of the receiving system at each of the m_{opt} stages of optimal search	$W_1=(\varepsilon/L)^{\frac{1}{2}}\times L$ $W_2=(\varepsilon/L)\times L=\varepsilon$
average localization time	$\frac{2}{\lambda}(\varepsilon/L)^{-\frac{1}{2}}$

Table-3. Required localization accuracy

$$\left(\frac{m}{m+1}\right)^{m(m+1)}\le(\varepsilon/L)\le\left(\frac{m-1}{m}\right)^{m(m-1)}$$

optimal number of stages m_{opt}	m
viewing windows of the receiving system at each of the m_{opt} stages of optimal search	$W_1=(\varepsilon/L)^{\frac{1}{m}}\times L$ $W_2=(\varepsilon/L)^{\frac{2}{m}}\times L$ … $W_i=(\varepsilon/L)^{\frac{i}{m}}\times L$ …
	$W_m=(\varepsilon/L)^{\frac{m}{m}}\times L=\varepsilon$
average localization time	$\frac{m}{\lambda}(\varepsilon/L)^{-\frac{1}{m}}$

III. TIME-OPTIMAL LOCALIZATION ALGORITHMS FOR MULTI-RECEIVER SYSTEMS

The problem considered in this section is to build an algorithm for a system with an arbitrary but fixed number of receivers that has the minimum (statistically) average time to reach the required localization accuracy. It seems unlikely to find the solution to the problem in general form (as already noted in the previous section). Therefore, when constructing optimal control algorithms for multi-receiving systems, we restrict ourselves to the frequent (although very important from a practical point of view) case when there is no prior information about the location of the source within the search interval, i.e. when it can be assumed that random source is uniformly distributed on the interval (0, *L*). The obvious advantage of such an algorithm: as a result of its application to search for a source having an arbitrary distribution (not uniform), the average localization time nevertheless remains constant and matches with the optimal localization time of a uniformly distributed source. Thus, knowledge of a prior information about the distribution of a pulsed object within a search interval, on the one hand, always leads to a complication of the optimal search algorithm, and, on the other hand, it always reduces the average search time.

Moreover, among all possible distributions of a pulsed object, the most time-consuming procedure is when the object has a uniform distribution. Tables 4-5 presents the example of optimal search parameters, which were calculated under the assumption that the receiving system has a fixed number of receivers $n\ge2$; the search is carried out on the interval (0, *L*); the required absolute localization accuracy is ε. The calculated parameters were: the optimal number of search stages; the size of the viewing windows at each stage; average time for an optimal search. When constructing the optimal localization procedure, it was considered acceptable and, moreover, it was assumed that not all pulses generated by a random source are recorded by the receiving system.

In this message Tables 4-5 contains only the final results, and all intermediate calculations are omitted.

Table-4. Required localization accuracy

$$\frac{1}{(2^n-1)^m}\le(\varepsilon/L)\le\frac{1}{(2^n-1)^{m-1}}\left(\frac{m-1}{m}\right)^{m-1}, (m=\overline{1,\infty})$$

optimal number of stages m_{opt}	m
view windows of the receiving system at each of the m_{opt} stages of optimal search	$W_1=L$ $W_2=\frac{1}{2^n-1}\times L$ … $W_m=\frac{1}{(2^n-1)^{m-1}}\times L$
average localization time	$\frac{m}{\lambda}$

Table-5. Required localization accuracy

$$\frac{e^{-1}}{(2^n-1)^{m_\infty}} \le (\varepsilon / L) \le \frac{e^{-1}}{(2^n-1)^{m_\infty - 1}}$$

optimal number of stages m_{opt} (asymptotic estimate of the number of search stages for $\varepsilon / L \to 0$)	$m_\infty^* \approx \frac{-\ln(\varepsilon / L)}{\ln(2^n - 1)} -$
view windows of the receiving system at each of the m_{opt} stages of optimal search	$W_1 = L$ $W_2 = \frac{1}{2^n - 1} \times L$ ... $W_i = \frac{1}{(2^n-1)^{i-1}} \times L$ ... $W_{m_\infty^*} = \frac{1}{(2^n-1)^{m_\infty^*-1}} \times L \approx$ $\approx \frac{2^n - 1}{(2^n-1)^{\frac{-\ln(\varepsilon / L)}{\ln(2^n-1)}}} \times L =$ $= (2^n - 1)\varepsilon$
average localization time	$\frac{m_\infty^*}{\lambda} \approx -\frac{\ln(\varepsilon / L)}{\lambda \ln(2^n - 1)}$

IV. RESULTS AND DISCUSSION

Described in this message one-dimensional optimal localization methods and algorithms can be used as a theoretical basis for constructing optimal algorithms in the multidimensional case. The example of such approach demonstrating a search scheme for an unknown pulsed-point source in two-dimensional case, is presented in Fig. 1.

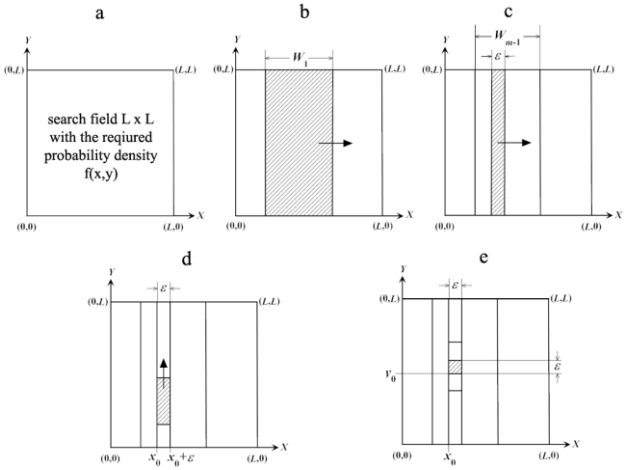

Fig 1. Search scheme for an unknown pulsed-point source in two-dimensional case: a – two-dimensional field with a given probability density distribution f (x, y); b - the first stage of localization along the X axis; c - the last stage of localization along the X axis; d - the first stage of localization along the Y axis; e - the last stage of localization along the Y axis.

So, we are given function *f(x, y)*– the a priori probability density of the distribution of a random source within the field $L \times L$ and ε – the required accuracy of localization of the pulsed source along both coordinates *x* and *y*. It is necessary to build an algorithm that allows for a minimum (statistically) time to find a fragment $\varepsilon \times \varepsilon$ inside the field that is guaranteed to contain a signal source. To do this, "roll up" two-dimensional probability density *f(x, y)* along the coordinate *y*: $f_1(x) = \int_0^L f(x.y)dy.$

Now we need to solve one-dimensional localization task with the following initial conditions: $f_1(x)$ – prior probability density of the pulsed source, *L* – the size of the search interval, ε – the required localization accuracy. As a result, we get the optimal search algorithm, which determines the number of scanning steps, the width of the receiver viewing window $W_i, i = \overline{1, m}$ for each of them (window's height, of course, should be equal to *L* at all stages of scanning) and, if necessary, the moments of switching the receiving system from one operating mode to another. Schematically, the first and *m*-th (last) stages of localization along the *X* axis are presented in Fig. 1b and 1c. At the *m*-th stage, which is carried out by a scanning window of size $\varepsilon \times L$ and completes the localization procedure by the *x* coordinate, it is necessary to remember the value x_0 corresponding to the position of the rectangular scanning aperture at the moment of fixing the last pulse. We will need this value when localizing according to the second coordinate *y*. Now, in order to perform the localization procedure of the source along the *Y* axis, it is necessary to correctly solve the one-dimensional search task inside the rectangular window $\varepsilon \times L$, which is limited by coordinates x_0 and $x_0 + \varepsilon$ along the *X* axis (see Fig. 1d). For this problem, the one-dimensional prior probability density of unknown source inside the scanned vertical ε-strip having a height *L* is determined by the relation

$$f_2(y) = \frac{\int_{x_0}^{x_0+\varepsilon} f(x.y)dx}{\int_0^L \left(\int_{x_0}^{x_0+\varepsilon} f(x.y)dx \right) dy}.$$

To clarify the position of the source along the *Y* axis, we reapply the procedure for optimal one-dimensional localization using probability density $f_2(y)$ for this. As a result, we get the final solution of the formulated two-dimensional problem in the form of a fragment of size $\varepsilon \times \varepsilon$ (see Fig. 1e), that is guaranteed to contain a pulsed source and has reference coordinates (x_0, y_0). Thus, to solve the two-dimensional localization problem, we needed to use the optimal one-dimensional search procedure twice.

V. CONCLUSION

Search strategies and calculated parameters presented in this paper open up the prospect of constructing optimal multi-stage localization algorithms for problems in which the probability density of a random pulsed source differs from a uniform one, and the search is performed by a multi-receiving system. In addition, of undoubted interest are problems that require the construction of physically realizable optimal algorithms for the localization of random pulsed sources with a piecewise constant probability density within the search segment ("physically realizable" algorithms are understood here as algorithms that can be implemented using simply connected scanning windows).

Another promising and poorly studied area of research is the constructing the optimal search procedures focused on the simultaneous localization of several random sources.

ACKNOWLEDGMENTS

This work was supported in part by the Russian Foundation for Basic Research (projects no 18-51-00001 and 19-01-00128), Basic Research program of Siberian Branch of the Russian Academy of Sciences (Federal Agency for Scientific Organizations project no. AAA-A17-117052410034-6).

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