Dominating Cocolouring of Graphs

M. Poobalaranjani, R.Pichailakshmi

Abstract: A k-cocolouring of a graph G is a partition of the vertex set into k subsets such that each set induces either a clique or an independent set in G. The chromatic number \( z(G) \) of a graph G is the least k such that G has a k-cocolouring of. A set \( S \subset V \) is a dominating set of G if for each \( u \in V - S \) there exists a vertex \( v \in S \) such that u is adjacent to v. The minimum cardinality of a dominating set in G is called the domination number and is denoted by \( \gamma(G) \). Combining these two concepts we have introduced new types of cocolouring viz, dominating cocolouring and \( \gamma \)-cocolouring. A dominating cocolouring of G is a cocolouring of G such that atleast one of the sets in the partition is a dominating set. Hence dominating cocolouring is a conditional cocolouring. The dominating co-chromatic number \( z_d(G) \) is the smallest cardinality of a dominating cocolouring of G (ie) \( z_d(G) = \min\{ k | G \text{ has a dominating cocolouring with } k \text{-colors} \} \).

Keywords: cocolouring, cochromatic number, dominating cocolouring, dominating chromatic number, dominating cocolorable graphs.

I. INTRODUCTION

Graphs considered in this paper are finite, simple and undirected. Given a simple graph \( G = (V,E) \), a subset \( W \) of V is called a clique provided that it induces a complete subgraph of G, and if \( W \) has cardinality \( k \) it is called a k-clique. Similarly, a subset \( U \) of V is called an independent set provided that it induces an empty subgraph of G, and if \( U \) has cardinality \( k \) it is called a k-independent set. The maximum cardinality of a clique in G is denoted \( \omega(G) \), whereas the maximum cardinality of an independent set is denoted \( \beta(G) \).

Vertex partition plays a major role in graph theory. New concepts and results were obtained by partitioning the vertex set and imposing on the sets. Two such partition is coloring are cocolouring. Let \( P = \{ V_1, V_2, \ldots, V_m \} \) be a partition of V. If each \( V_i \) is an independent set, then P is called an n-coloring of G, while if each \( V_i \) is an independent set or a clique, then P is called an n-cocoloring of G. Recall that the chromatic number \( x(G) \) is the minimum for which there exists n-coloring. The cochromatic number \( z(G) \) is the minimum positive integer m for which there exists an m-cocoloring of G. Hence the cochromatic number \( z(G) \) can be equivalently defined as \( z(G) = \min\{x + i, k, l \geq 0, G \text{ is } (k,l) - \text{colorable} \} \). The cocolouring does not restrict the number of independent sets and cliques, any proper colouring is a cocolouring and a clique partition is also a cocolouring. This yields that \( z(G) \leq \min\{\gamma(G), \beta(G)\} \). This definition was introduced by Lesniak and Straight[4]. Later P.Erdos and J.Gimbel [3] gave further results on chromatic number in 1990.

The study of domination in graphs was further developed in the late 1950’s and 1960’s, beginning with Claude Berge [1] in 1958. Berge wrote a book on graph theory, in which he introduced the “coefficient of external stability,” which is now known as the domination number of a graph. OysteinOre [5] introduced the terms “dominating set” and “domination number” in his book on graph theory which was published in 1962. The domination in graphs has been studied extensively and several additional research papers have been published on this topic. A set \( S \subset V \) is a dominating set of G if for each \( u \in V - S \), there exists a vertex \( v \in S \) such that u is adjacent to v. Combining these two concepts we have introduced new types of cocolouring, dominating cocolouring, as the name suggests cocolouring contain a dominating set, the former insists that the dominating set induces either a null graph or a complete graph.

II. PRIOR RESULTS

Proposition 2.1: If G is a graph other than \( K_2 \) with \( \omega(G) < 3 \), then \( \chi(G) = z(G) \).

Definition 2.2: A graph G is critically cochromatic if \( z(G-v) < z(G) \) for each vertex of G.

III. DOMINATING COCOLORING OF GRAPH

Definition 3.1: A dominating cocolouring of G is a cocolouring of G such that atleast one of the sets in the partition is a dominating set. Hence dominating cocolouring is a conditional cocolouring. The dominating co-chromatic number \( z_d(G) \) is the smallest cardinality of a dominating cocolouring of G (ie) \( z_d(G) = \min\{ k | G \text{ has a dominating cocolouring with } k \text{-colors} \} \).

Example 3.2:

Black vertices in (b) and (c) of figure 3.1 denote the dominating set of G respectively. Grey vertices denote the clique and white vertices denote the independent set.

![Figure 3.1](image)

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Observation 3.3: For any graph $G$, 
   i) dominating cocoloring exists;  
   ii) $z(G) \leq z_2(G)$;  
   iii) $z_2(G) = 1$ if and only if $G = K_n$ or $nK_1$;  
   iv) If any minimum cocoloring contains a maximal independent set, then $z(G) = z_2(G)$. 

Proof: 
   i) Every graph has a maximal independent set. A maximal independent set is a dominating set and hence any cocoloring which contains a maximal independent is a dominating cocoloring.  
   ii) Follows from the fact that a dominating cocoloring is a conditional cocoloring.  
   iii) Trivial.  
   iv) A maximal independent set is a dominating set. 

Definition 3.4: If $G$ is a graph with $z(G) = z_2(G)$, then $G$ is called a dominating cocolorable graph. 

Example 3.5: 

(a) Dominating cocolorable graph with $z = 2$ and $z_2 = 3$  
(b) Graph which is not dominating cocolorable graph with $z = z_2 = 2$ 

Figure 3.2 

Proposition 3.6: Bipartite graphs are dominating cocolorable graphs. 

Proof: Let $G$ be a bipartite graph. If $G = K_{2,2}$, then $z(G) = z_2(G) = 1$ and the result is proved. Suppose $G \neq K_{2,2}$, then the vertices of $G$ can be partitioned into two independent sets and hence $z_2(G) = 2$. 

If $G$ is a connected graph, then both the independent sets are dominating. 

If $G$ is a disconnected graph without isolated vertices, then again the two independent sets are independent dominating sets. 

If $G$ has isolated vertices, then the independent set containing all the isolated vertices is a dominating set. Hence $z_2(G) = 2$. 

The following theorem and proposition give sufficient conditions for a graph to be a dominating cocolorable graph. 

Theorem 3.7: Let $G$ be a graph such that $\chi(G) = z(G)$. Then $G$ is a dominating cocolorable graph. 

Proof: WLG let $G$ be a connected graph. Let $\chi(G) = k$ and $S = \{I_1, I_2, ..., I_k\}$ be a chromatic partition of $G$. If $I_1$ is a dominating set for some $i$, then $S$ is a dominating cocoloring of $G$. Hence $z_2(G) = z_2(S)$ over the result follows. 

Suppose $I_1$ is not a dominating set for each $i$. Let $I_2 = I'_2 \cup I''_2$, where $I'_2 = \{x \in I_2: N(x) \cap I_1 = \emptyset\}$ and $I''_2 = \{x \in I_2: N(x) \cap I_1 \neq \emptyset\}$. 

Hence $I'_2$ consist of all vertices of $I_2$ which are not dominated by $I_1$ and $I''_2$ consists of all vertices of $I_2$ when are not dominated by $I_1$. 

Claim: $I''_2 \neq \emptyset$. 

If $I''_2 = \emptyset$ then $I_2 = I'_2$. Hence $I_2$ has no neighbors us $I_1$ and this leads to $I_1 \cup I_2$ is an independent set. Hence let $S$, $I_1$, and $I_2$ can be merged who a single set and thus $\chi(G)$ can be reduced to $k - 1$, a contradiction to $\chi(G) = k$. Hence the claim. 

To define sets recursively, we rename $I_1$ as $I_{1,1}$ and let $I_{1,2} = I'_2 \cup I''_2$. 

Clearly $I_{1,2}$ is an independent let and dominate $I_{1,2}$. 

Now define recursively the following sets. For $3 \leq j \leq k$, $I_j = \{x \in I_j: N(x) \cap I_{j-1} = \emptyset\}$ and $I_j'' = \{x \in I_j: N(x) \cap I_{j-1} \neq \emptyset\}$. 

Finally, let $I_k = I_k''$. 

Further are include the claim, $I_4 = I_1 \cup I_2 \cup ... \cup I_k$ is obtained. It can be clearly seen that $I_{1,2}, I_{2,2}, I_{3,2}, I_{4,2}$ one all independent sets and $I_{1,2}$ is dominated by $I_{1,1}$ and hence by $I_1$. 

Further $I_j'' \neq \emptyset \ \forall j$. Otherwise as it can of $I_j'' = \emptyset$, $\chi(G)$ can be reduced and hence a contradiction on see. Thus $S' = \{I_1', I_2', ..., I_k'\}$ is a coloring of $G$. Hence $I_1'$ is a dominating set. Hence the theorem. 

Figure 3.3 

Proposition 3.8: If $G$ is a graph with $\omega(G) < 3$, then $G$ is a dominating cocolorable graph. 

Proof: If $G = K_2$, then $z(G) = z_2(G) = 1$. Now suppose $G \neq K_2$. Then by proposition 2.1, $\chi(G) = z(G)$. The result follows from theorem 3.7. 

Proposition 3.9: Let $G$ be a graph with a dominating vertex. Then $z(G) = z_2(G)$. 

Proof: Let $u$ be a dominating vertex and $P$ a cocoloring of $G$. In $P$, any set containing $u$ is a dominating set and hence $P$ is a dominating cocoloring of $G$. Since $P$ is arbitrary, every cocoloring of $G$ is a dominating cocoloring of $G$. Hence, $z(G) = z_2(G)$ and the theorem is a dominating cocolorable graph. 

Proposition 3.10: For any graph $G$, $z(G) \leq z_2(G) \leq z(G) + 1$. 

Proof: Let $S = \{I_1, I_2, ..., I_k\}$ be a minimal cocoloring of $G$ where $I_j$'s one independent sets and $C_j$'s are cliques. Let $D$ be a maximal independent set. If $D \cap C_j = \emptyset$ then let $C_j$'s as it is. If $S \cap C_j = \emptyset$ for some $j$, then $|D \cap C_j| = 1$ as $D$ is an independent set. 

Let $D \cap C_j = \{u\}$. If $|C_j| = 1$, then $C_j = \emptyset$. If $|C_j| > 1$, then $C_j = C_j - \{u\}$. 

Replace $C_j$ by $C_j$ in $S$. Repeat this for all cliques in $S$. Suppose $D = I_1$, then $S$ is a dominating cocoloring $z = z_2$. Otherwise $D \neq I_1$, then leave $I_1$. Suppose $I_1 \subset D$ then remove $I_1$ from $S$ and $I'_1 = \emptyset$. 

If $D \cap I_1 \neq \emptyset$ then let $I_2 = I_1 - D \cap I_1$. Repeat this for $I_1$'s and call this new $D$ as $D'$. Let $S'$ contain $I_1$'s, $C_j$'s and $D'$ or $S' = \{I_1', I_2', ..., I_k', C_1', C_2', ..., C_l', D'\}$ in which $I_1'$ or $C_j'$ may be empty. Here $S'$ contain atmost $t + k + 1$ non-empty set in which $D'$ is an independent dominating set. 

$S'$ is a dominating cocoloring. 

$z(G) \leq z(S') = t + k + 1 = z(G) + 1$. If $D = I_1$ for some $i$, then $z_2(G) = z(S)$ always. Suppose $D \neq I_1$ for some $i$. If $D \cap I_1 = \emptyset$ then $|D \cap I_1| = 1$ as $D$ is an independent set. Hence in both cases $D \cap I_1 = \emptyset$. 

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If \( D \cap I_i = \emptyset \) then let \( I'_i = I_i \). Suppose if \( D \cap I'_i \neq \emptyset \). If \( I'_i \subseteq D \) then \( I'_i = I_i \) and if \( I'_i \cap D \) let \( I'_i' = I_i - D \cap I_i \). In all cases \( D \cup I_i = \emptyset \). If \( D \cap C_j = \emptyset \) then \( C'_j = C_j \). If \( D \cap C'_j = 0 \) then \( |D \cup C'_j| = 1 \). Let \( C'_j = C_j - D \cap C_j \) Hence \( D \cap C'_j = \emptyset \).

Let \( S' = \{I'_{i_1}, I'_{i_2}, \ldots, I'_{i_n}, C'_1, C'_2, \ldots, C'_{n'}, D \} \). Clearly \( S' \) is a dominating cocoloring of \( G - S \) contain atmost one set may be empty. Hence \( z(G) \leq z_d(G) \leq z(G) + 1 \).

**CONCLUSION**

In this paper, a new cocoloring called dominating cocoloring and its corresponding parameter are defined. We have proved that this cocoloring exists for all graphs. Accordingly, a new class of graph called dominating cocolorable graph is defined. Further, some sufficient conditions for a graph to be a dominating cocolorable graph are proved.

**REFERENCES**


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