Dominating Cocoloring of Graphs

M. Poobalaranjani, R.Pichailakshmi

Abstract: *A k-cocolouring of a graph G is a partition of the vertex set into k subsets such that each set induces either a clique or an independent set in G. The cochromatic number $z(G)$ of a graph G is the least k such that G has a k-cocolouring of G. A set $S \subseteq V$ is a dominating set of G if for each $u \in V - S$, there exists a vertex $v \in S$ such that u is adjacent to v. The minimum cardinality of a dominating set in G is called the domination number and is denoted by $\gamma(G)$. Combining these two concepts we have introduces two new types of cocoloring viz, dominating cocoloring and γ-cocoloring. A dominating cocoloring of G is a cocoloring of G such that atleast one of the sets in the partition is a dominating set. Hence dominating cocoloring is a conditional cocoloring. The dominating co-chromatic number $z_d(G)$ is the smallest cardinality of a dominating cocoloring of G .(ie) $z_d(G) = min\{ k|G$ has a dominating cocoloring with k-colors$\}$.*

Keywords : *cocolouring, cochromatic number, dominating cocoloring, dominating cochromatic number, dominating cocolorable graphs.*

I. INTRODUCTION

Graphs considered in this paper are finite, simple and undirected. Given a simple graph $G = (V,E)$, a subset W of V is called a clique provided that it induces a complete subgraph of G, and if W has cardinality k it is called a k-clique. Similarly, a subset U of V is called an independent set provided that it induces an empty subgraph of G, and if U has cardinality k it is called a k-independent set. The maximum cardinality of a clique in G is denoted $\omega(G)$, whereas the maximum cardinality of an independent set is denoted $\beta(G)$.

Vertex partition plays a major role in graph theory. New concepts and results were obtained by partitioning the vertex set and imposing on the sets. Two such partition is coloring are cocoloring. Let $P = \{V_1, V_2, \ldots, V_n\}$ be a partition of V. If each V_i is an independent set, then P is called an n-coloring of G, while if each V_i is an independent set or a clique, then P is called an n-cocoloring of G.Recall that the chromatic number $x(G)$ is the minimum for which there exists n-coloring. The cochromatic number $z(G)$ is the minimum positive integer m for which there exists an m-cocoloring of G. Hence the co chromatic number $z(G)$ can be equivalently defined as $z(G) = \min\{k + l, k, l \geq 0, G\ is\ (k, l) - \text{colorable}\}$. The cocolouring does not restrict the number of independent sets and cliques, any proper colouring is a cocolouring and a clique partition is also a cocolouring. This yields that $z(G) \leq min\{\chi(G), \theta(G)\}$.This definition was introduced by Lesniak and Straight[4].Later P.Erdos and J.Gimbel [3] gave further results on cochromatic number in 1990.

The study of domination in graphs was further developed in the late 1950's and 1960's, beginning with Claude Berge [1] in 1958. Berge wrote a book on graph theory, in which he introduced the1"coefficient of external stability," which is now known as the domination number of a graph. OysteinOre [5] introduced the terms "dominating set" and "domination number" in his book on graph theory which was published in 1962.The domination in graphs has been studied extensively and several additional research papers have been published on this topic. A set $S \subseteq V$ is a ***dominating set*** of G if for each $u \in V - S$, there exists a vertex $v \in S$ such that u is adjacent to v. Combining these two concepts we have introduces a new types of cocoloring ,dominating cocoloring .as the name suggests cocoloring contain a dominating set,the former insists that the dominating set induces either a null graph or a complete graph.

II. PRIOR RESULTS

Proposition 2.1: If G is a graph other than K_2 with $\omega(G) < 3$, then $\chi(G) = z(G)$.

Definition 2.2: A graph G is critically cochromatic if z(G-v) < z(G) for each vertex of G.

III. DOMINATING COCOLORING OF GRAPH

Definition 3.1: A dominating cocoloring of G is a cocoloring of G such that atleast one of the sets in the partition is a dominating set.

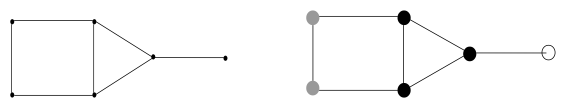
Hence dominating cocoloring is a conditional cocoloring.

The dominating co-chromatic number $z_d(G)$ is the smallest cardinality of a dominating cocoloring of G.

(ie) $z_d(G) = \min\{ k|G$ has a dominating cocoloring with k-colors$\}$.

Example 3.2:

Black vertices in (b) and (c) of figure 3.1 denote the dominating set of G respectively. Grey vertices denote the clique and white vertices denote the independent set.



(a) Graph G (b)Dominating coloring of G

Figure 3.1

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M.Poobalaranjani[1], PG & Research Department of Mathematics Seethalakshmi Ramaswami College, Tiruchirappalli, Tamil Nadu, India.Email.Id: mpranjani@hotmail.com

R.Pichailakshmi[2], PG & Research Department of Mathematics Seethalakshmi Ramaswami College, Tiruchirappalli, Tamil Nadu, India. Email. Id:laxmirajesh1806@gmail.com

Observation 3.3: For any graph G,

i) dominating cocoloring exists;

ii) $z(G) \le z_d(G)$;

iii) $z_d(G) = 1$ if and only if $G = K_n$ or nK_1;

iv) If any minimum cocoloring contains a maximal independent set, then $z(G) = z_d(G)$.

Proof:

i) Every graph has a maximal independent set. A maximal independent set is a dominating set and hence any cocoloring which contains a maximal independent is a dominating cocoloring.

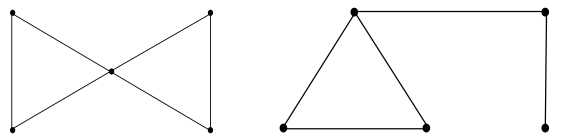
ii) Follows from the fact that a dominating cocoloring is a conditional cocoloring.

iii) Trivial.

iv) A maximal independent set is a dominating set.

Definition 3.4: If G is a graph with $z(G) = z_d(G)$, then G is called a dominating cocolorable graph.

Example 3.5:



(a) **(b)**

(a) Dominating cocolorable graph with $z = 2$ and $z_d = 3$.

(b) Graph which is not dominating cocolorable graph with $z = z_d = 2$

Figure 3.2

Proposition 3.6: Bipartite graphs are dominating cocolorable graphs.

Proof: Let G be a bipartite graph. If G = K_2, then z(G) = z_d(G) = 1 and the result is proved. Suppose G ≠ K_2. Then the vertices of G can be partitioned into two independent sets and hence $z(G) = 2$.

If G is a connected graph, then both the independent sets are dominating.

If G is a disconnected graph without isolated vertices, then again the two independent sets are independent dominating sets.

If G has isolated vertices, then the independent set containing all the isolated vertices is a dominating set. Hence $z_d(G) = 2$

The following theorem and proposition give sufficient conditions for a graph to be a dominating cocolorable graph.

Theorem 3.7: Let G be a graph such that χ(G) = z(G). Then G is a dominating cocolorable graph.

Proof: WLG let G be a connected graph. Let $\chi(G) = k$ and $S = \{I_1, I_2, \dots, I_k\}$ be a chromatic partition of G. If I_i is a dominating set for some i, then S is a dominating cocoloring of G. Hence $z(G) = z_d(G)$ over the result follows. So suppose I_i is not a dominating set for each i. Let $I_2 = I_2' \cup I_2''$, where $I_2' = \{x \in I_2 : N(x) \cap I_1 = \emptyset\}$ and $I_2'' = \{x \in I_2 : N(x) \cap I_1 \neq \emptyset\}$. Hence I_2' consist of all vertices of I_2 which are not dominated by I_1 and I_2'' consists of all vertices of I_2 when are not dominated by I_1.

Claim: $I_2'' \neq \emptyset$.

If $I_2'' = \emptyset$ then $I_2 = I_2'$. Hence I_2 has no neighbours us I_1 and this leads to $I_1 \cup I_2$ is an independent set. Hence let S, I_1 and I_2 can be merged who a single set and thus $\chi(G)$ can be reduced to $k-1$, a contradiction to $\chi(G) = k$. Hence the claim.

To define sets recursively, we rename I_1 as $I_1{}'$ and let $I_2^* = I_2^*$ and $I_1^2 = I_1' \cup I_2'$.

Clearly I_1^2 is an independent let and dominate I_2^*.

Now define recursively the following sets. For $3 \le j \le k, I_j' = \{x \in I_j : N(x) \cap I_1^{j-1} = \emptyset\}$, $I_1'' = \{x \in I_j : N(x) \cap I_1^{j-1} \neq \emptyset\}$, $I_1^* = I_1^{j-1} \cup I_j'$, $I_j^* = I_j''$.

Finally, let $I_1^* = I_1^k$.

Further are include the claim, $I_1 = I_1' \subset I_1^2 \subset \dots \subset I_1^k = I_1^*$ is obtained. It can be clearly seen that I_1', I_1^2, I_1^k one all independent sets and I_j^* is dominated by I_1^{j-1} and hence by I_1^*. Further $I_j^* \neq \emptyset\ \forall j$. Otherwise as it can of $I_2''(= I_2^*)$, $\chi(G)$ can be reduced and hence a contradiction on see. Thus $S^* - \{I_1^*, I_2^*, \dots, I_k^*\}$ is a coloring of G. Hence I_1^* is a dominating set. Hence the theorem.

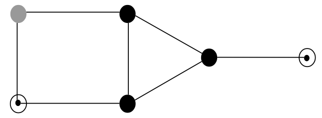


Figure 3.3

Proposition 3.8: If G is a graph with $\omega(G) < 3$, then G is a dominating cocolorable graph.

Proof: If G = K_2, then z(G) = z_d(G) = 1. Now suppose G ≠ K_2. Then by proposition 2.1, χ(G) = z(G). The result follows from theorem 3.7.

Proposition 3.9: Let G be a graph with a dominating vertex. Then z(G) = z_d(G).

Proof: Let u be a dominating vertex and $\mathcal{P}$ a cocoloring of G. In $\mathcal{P}$, any set containing u is a dominating set and hence $\mathcal{P}$ is a dominating cocoloring of G. Since $\mathcal{P}$ is arbitrary, every cocoloring of G is a dominating cocoloring of G. Hence, z(G) = z_d(G). and the theorem is s a dominating cocolorable graph.

Proposition 3.10: For any graph $G, z(G) \le z_d(G) \le z(G) + 1$

Proof: Let $S = \{I_1, I_2, \dots, I_k, C_1, C_2, \dots, C_l\}$ be a minimum cocoloring of G where I_i's one independent sets and C_j's are cliques. Let D be a maximal independent set. If $D \cap C_j = \phi$ then leave C_j's as it is. If $S \cap C_j = 0$ for some j then $|D \cap C_j| = 1$ as D is an independent set.

Let $D \cap C_j = \{u\}$. If $|C_j| = 1$, then $C_j' = \phi$. If $|C_j| > 1$ then $C_j' = C_j - \{u\}$.

Replace C_j by C_j' in S. repeat this for all cliques in S. Suppose $D = I_i$ then S is a dominating cocoloring $z = z_d$. Otherwise $D \neq I_i$ then leave I_i. Suppose $I_1 \subset D$ then remove I_i from S and $I_1' = \phi$.

If $D \cap I_j \neq \phi$ then let $I_j' = I_j - D \cap I_j$. repeat this for I's and call this new D as D'. Let S' contain I_1's, C_j's and D' or $S' = \{I_1', I_2', \dots, I_k', C_1', C_2', \dots, C_l', D'\}$ in which I_i' or C_j' may be empty. Here S' contain atmost $l + k + 1$ non-empty set in which D' is an independent dominating set. $\therefore S'$ is a dominating cocoloring. $\therefore z_d(G) \le |S'| = l + k + 1 = z(G) + 1$. If $D = I_i$ for some i then $z_d(G) = z(G)$. Suppose $D \neq I_i$ for some i. If $D \cap I_i = \phi$ then $|D \cap I_i| = 1$ as D is an independent set. Hence in both cases $D \cap I_i' = \phi$.

If $D \cap I_i = \phi$ then let $I_i' = I_i$. Suppose if $D \cap I_i' \neq \phi$. If $I_i \subset D$ then $I_i' = \phi$ and if $I_i \not\subset D$ let $I_i' = I_i - D \cap I_i$. In all cases $D \cap I_i' = \phi$. If $D \cap C_j = \phi$ then $C_j' = C_j$. If $D \cap C_j \neq 0$ then $|D \cap C_j| = 1$. Let $C_j' = C_j - D \cap C_j$. Hence $D \cap C_j' = \phi$.

Let $S' = \{I_1', I_2', \dots, I_n', C_1', C_2', \dots, C_s', D\}$. Clearly S' is a dominating cocoloring of $G - S$ contain atmost one set may be empty. Hence $z(G) \leq z_d(G) \leq z(G) + 1$.

CONCLUSION

In this paper, In this paper In this paper, a new cocoloring called dominating cocoloring and its corresponding parameter are defined. We have proved that this cocoloring exists for all graphs. Accordingly, a new class of graph called dominating cocolorable graph is defined. Further, some sufficient conditions for a graph to be a dominating cocolorable graph are proved.

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AUTHORS PROFILE

First Author Poobalaranjani has received her M.Sc., &M.Phil Mathematics from the Bharathidasan University, Tiruchirappali, India in 1984 and 1990 respectively. In 2003, she received M.C.A degree from the Bharathidasan University. She did her research work in the National Institute of Technology, Tiruchirppali (NITT) and received her Ph.D in 2006 from Bharathidasan University. Her area of interest is Domination and coloring in Graph Theory. She joined as an Assistant Professor in the Department of Mathematics, Seethalakshmi Ramaswami College, Tiruchirappali, India in 1986. Since then she was with the department. In 2015, she became the head of the Department of Mathematics.

Second Author R.Pichailakshmi received her M.Sc., and M.Phil degrees from the Bharathidasan University, Trichy, India, in 2005, and 2006, respectively. She joined as a lecture in Department of Mathematics, Seethalakshmi Ramaswami College, Trichy, India in 2009. Her main area of research interest is Graph theory, particularly in graph partitioning. Currently she is doing her Ph.D under the guidance of Dr. M. Poobalaranjani.