Complex Number and its Conjugate in Continued Fraction

A. Gnanam, S. Krithika

ABSTRACT: For any equation complex roots occur in pairs. For finding a root of the equation in continued fractions algorithms are available. But to get complex roots in continued fraction no such procedure so far we have. Here, in this paper if $\alpha + i\beta$ has a representation in continued fractions we try to find its conjugate $\alpha - i\beta$ in terms of continued fractions. This will be useful in finding complex roots of a quadratic equations in continued fraction.

KEYWORDS: Continued Fraction, Complex Continued Fraction, Complex numbers.

I. INTRODUCTION

An expression of the form

$$a_0 + \cfrac{e_0}{a_1 + \cfrac{e_1}{a_2 + \cfrac{e_2}{a_3 + \cfrac{e_3}{\ddots + \cfrac{e_n}{a_n + \ddots}}}}}$$

Where $a_0, a_1, a_2, a_3, \cdots$ are in $Z(i)$ and e_i's are units of complex numbers. Therefore $e_k \in \{1,-1,i,-i\}, k = 1,2,3,\ldots$ is known as a complex continued fraction.**[1,2,3,4]**

Therefore $a_0 + \frac{e_1}{a_1 +}\frac{e_2}{a_2 +}\frac{e_3}{a_3 +}\cdots\frac{e_n}{a_n}$ is known as finite complex continued fraction and an expression

$$a_0 + \frac{e_1}{a_1 +}\frac{e_2}{a_2 +}\frac{e_3}{a_3 +}\cdots\frac{e_n}{a_n +}\frac{e_{n+1}}{a_{n+1} +}\cdots$$ is known

as an infinite complex continued fraction. In a finite or infinite complex continued fraction $a_0, a_1, a_2, a_3, \cdots$ are called the partial quotients where as $e_1, e_2, e_3, \cdots$ are known as the partial numerators.

1. ***Algorithm of complex continued fraction:*****[3,4]**

Let $x \in C$. Suppose we wish to find continued fraction expansion of X, take $x_0 = \text{Re}(x) + i\,\text{Im}(x)$.

Let $a = \text{Re}(x) - [\text{Re}(x)]$ and $b = \text{Im}(x) - [\text{Im}(x)]$ so that $a < 0$ and $b < 0$.

Now $[x_0] = [\text{Re}(x)] + i[\text{Im}(x)] + \alpha$, a floor function.

Where $$\alpha = \begin{cases} 0 & if\ a + b < 0 \\ 1 & if\ a + b \geq 1\ with\ a \geq b \\ i & if\ a + b \geq 1\ with\ a < b \end{cases}$$

Set $a_0 = [x_0]$. Then $x_1 = \dfrac{1}{x_0 - [x_0]}$ and $[x_1] = [\text{Re}(x_1)] + i[\text{Im}(x_1)] + \alpha$.

Again set $a_1 = [x_1]$. Then $x_2 = \dfrac{1}{x_1 - [x_1]}$ and $[x_2] = [\text{Re}(x_2)] + i[\text{Im}(x_2)] + \alpha$, $a_2 = [x_2]$ Then $x_3 = \dfrac{1}{x_2 - [x_2]}$ and $[x_3] = [\text{Re}(x_3)] + i[\text{Im}(x_3)] + \alpha$, $a_3 = [x_3]$ .... $a_{k-1} = [x_{k-1}]$.

Then $x_k = \dfrac{1}{x_{k-1} - [x_{k-1}]}$ and $[x_k] = [\text{Re}(x_k)] + i[\text{Im}(x_k)] + \alpha$ .and $a_k = [x_k]$.

The algorithm terminates if the complex continued fraction is finite otherwise it is no terminating.

Theorem

If the complex continued fraction of

$$\alpha + i\beta = \langle \alpha_0 + i\beta_0, \alpha_1, \alpha_2 + i\beta_2, \alpha_3 + i\beta_3, \alpha_4 + i\beta_4, \ldots \rangle$$

such that $\alpha, \beta \in Q$ the set of all rational numbers then the complex continued fraction of

$$\alpha - i\beta = \langle \alpha_0 - i\beta_0, \alpha_1 - i, \alpha_2 + i\beta_2, \alpha_3 + i\beta_3, \alpha_4 + i\beta_4, \ldots \rangle$$ where

α_i's, β_i's are integers.

Proof:

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A.Gnanam, Assistant professor ,Department of Mathematics, Government Arts College, Trichy-22. Email: gnaanam@yahoo.com

S.Krithika, Assistant professor ,Department of Mathematics, Seethalakshmi Ramaswami College, Trichy-2 Email:krithikasrc@gmail.com

Taking $\alpha_k + i\beta_k = c_k, k = 0,1,2,3...,$

$\alpha + i\beta = \langle c_0, c_1, c_2, c_3, c_4, ...\rangle$

Then the successive convergent of this complex continued fraction is denoted by $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \frac{p_4}{q_4}, ...$ in which either the third or fourth convergent gives the desired solution.

Now $\frac{p_0}{q_0} = c_0, \quad \frac{p_1}{q_1} = \langle c_0, c_1\rangle = \frac{c_0c_1 + 1}{c_1}$

$$\frac{p_2}{q_2} = \langle c_0, c_1, c_2\rangle = \frac{c_0c_1c_2 + c_0 + c_2}{c_1c_2 + 1}$$

$$\frac{p_3}{q_3} = \langle c_0, c_1, c_2, c_3\rangle = \frac{c_0c_1c_2c_3 + c_0c_1 + c_0c_3 + c_2c_3 + 1}{c_1c_2c_3 + c_1 + c_3}$$

$$\frac{p_4}{q_4} = \langle c_0, c_1, c_2, c_3, c_4\rangle$$

$$= \frac{c_0c_1c_2c_3c_4 + c_0c_1c_2 + c_0c_1c_4 + c_0c_3c_4 + c_2c_3c_4 + c_0 + c_2 + c_4}{c_1c_2c_3c_4 + c_1c_2 + c_1c_4 + c_3c_4 + 1}$$

Taking $c_k = \alpha_k + i\beta_k, k = 0,1,2,3...,$ then

For every $i, j = 0,1,2,3,...$

$$c_i c_j = (\alpha_i\alpha_j - \beta_i\beta_j) + i(\alpha_i\beta_j - \beta_i\alpha_j)$$

For every $i, j, l = 0,1,2,3,...$

$$c_i c_j c_l = (\alpha_i\alpha_j\alpha_l - \alpha_i\beta_j\beta_l - \beta_i\alpha_j\beta_l - \beta_i\beta_j\alpha_l) + i(\beta_i\alpha_j\alpha_l - \beta_i\beta_j\beta_l + \alpha_i\beta_j\alpha_l + \alpha_i\alpha_j\beta_l)$$

For every $i, j, l, m = 0,1,2,3,...$

$$c_i c_j c_l c_m = (\alpha_i\alpha_j\alpha_l\alpha_m - \alpha_i\beta_j\beta_l\alpha_m - \beta_i\alpha_j\beta_l\alpha_m - \beta_i\beta_j\alpha_l\alpha_m - \beta_i\alpha_j\alpha_l\beta_m + \beta_i\beta_j\beta_l\beta_m - \alpha_i\beta_j\alpha_l\beta_m - \alpha_i\alpha_j\beta_l\beta_m) + i(\beta_i\alpha_j\alpha_l\alpha_m - \beta_i\beta_j\beta_l\alpha_m + \alpha_i\beta_j\alpha_l\alpha_m + \alpha_i\alpha_j\beta_l\alpha_m + \alpha_i\alpha_j\alpha_l\beta_m - \alpha_i\beta_j\beta_l\beta_m - \beta_i\alpha_j\beta_l\beta_m - \beta_i\beta_j\alpha_l\beta_m)$$

For every $i, j, l, m, n = 0,1,2,3,...$

$$c_i c_j c_l c_m c_n = (\alpha_i\alpha_j\alpha_l\alpha_m\alpha_n - \alpha_i\alpha_j\beta_l\beta_m\alpha_n - \alpha_i\beta_j\alpha_l\beta_m\alpha_n - \alpha_i\beta_j\beta_l\alpha_m\alpha_n - \alpha_i\beta_j\alpha_l\alpha_m\beta_n + \alpha_i\beta_j\beta_l\beta_m\beta_n - \alpha_i\alpha_j\beta_l\alpha_m\beta_n - \alpha_i\alpha_j\alpha_l\beta_m\beta_n - \beta_i\beta_j\alpha_l\alpha_m\alpha_n + \beta_i\beta_j\beta_l\beta_m\alpha_n - \beta_i\alpha_j\beta_l\alpha_m\alpha_n - \beta_i\alpha_j\alpha_l\beta_m\alpha_n - \beta_i\alpha_j\alpha_l\alpha_m\beta_n + \beta_i\alpha_j\beta_l\beta_m\beta_n + \beta_i\beta_j\alpha_l\beta_m\beta_n + \beta_i\beta_j\beta_l\alpha_m\beta_n) + i(\alpha_i\beta_j\alpha_l\alpha_m\alpha_n - \alpha_i\beta_j\beta_l\beta_m\alpha_n + \alpha_i\alpha_j\beta_l\alpha_m\alpha_n + \alpha_i\alpha_j\alpha_l\beta_m\alpha_n + \alpha_i\alpha_j\alpha_l\alpha_m\beta_n - \alpha_i\alpha_j\beta_l\beta_m\beta_n - \alpha_i\beta_j\alpha_l\beta_m\beta_n - \alpha_i\beta_j\beta_l\alpha_m\beta_n + \beta_i\alpha_j\alpha_l\alpha_m\alpha_n - \beta_i\alpha_j\beta_l\beta_m\alpha_n - \beta_i\beta_j\alpha_l\beta_m\alpha_n - \beta_i\beta_j\beta_l\alpha_m\alpha_n - \beta_i\beta_j\alpha_l\alpha_m\beta_n + \beta_i\beta_j\beta_l\beta_m\beta_n - \beta_i\alpha_j\beta_l\alpha_m\beta_n - \beta_i\alpha_j\alpha_l\beta_m\beta_n)$$

II. RESULT ANALYSIS:

Consider the complex numbers $\alpha \pm i\beta = 1.1 \pm 0.9950\, i$

Using Continued fraction algorithm:
Continued fraction of the complex number
$\alpha + i\beta = 1.1 + 0.9950\, i$

Take $x_0 = 1.1 + 0.9950i$

Then $a = 1.1 - (1) = 0.1000$ and $b = 0.9950 - (0) = 0.9950$

Therefore $[x_0] = 1 + 0 + i = 1 + i \Rightarrow a_0 = 1 + i$.

Now

$$x_1 = \frac{1}{[(1.1 + 0.9950i) - (1 + i)]} \Rightarrow x_1 = 9.9751 + 0.4988i$$

Round off the imaginary part to 0.5 , $x_1 = 9.9751 + 0.5i$

Then $a = 9.9751 - (9) = 0.9751$ and $b = 0.5 - (0) = 0.5$

Therefore $[x_1] = 9 + 0 + 1 = 10 \Rightarrow a_1 = 10$.

Again

$$x_2 = \frac{1}{[(9.9751 + 0.5i) - (10)]} \Rightarrow x_2 = -0.0993 - 1.9951i$$

Then $a = -0.0993 - (-1) = 0.0007$ and $b = -1.9951 - (-2) = 0.0049$

Therefore $[x_2] = -1 - 2i + 0 = -1 - 2i \Rightarrow a_2 = -1 - 2i$.

$$x_3 = \frac{1}{[(-0.0993 - 1.9951i) - (-1 - 2i)]} \Rightarrow x_3 = 1.1102 - 0.0060i$$

Then $a = 1.1102 - (1) = 0.1102$ and $b = -0.0060 - (-1) = 0.9940$

Therefore $[x_3] = 1 - i + i = 1 \Rightarrow a_3 = 1$.

$$x_4 = \frac{1}{[(1.1102 - 0.0060i) - (1)]} \Rightarrow x_4 = 9.0476 + 0.4926i$$

Then $a = 9.0476 - (9) = 0.0476$ and $b = 0.4926 - (0) = 0.4926$

Therefore $[x_4] = 9 + 0 + 0 = 9 \Rightarrow a_4 = 9$.

$$x_5 = \frac{1}{[(9.0476 + 0.4926i) - (9)]} \Rightarrow x_5 = 0.1943 - 2.0113i$$

Then $a = 0.1943 - (0) = 0.1943$ and $b = -2.0113 - (-3) = 0.9887$

Therefore $[x_5] = 0 - 3i + i = -2i \Rightarrow a_5 = -2i$

$$x_6 = \frac{1}{[(0.1943 - 2.0113i) - (-2i)]}$$

$$\Rightarrow x_6 = 5.1293 + 0.2983i$$

Then $a = 5.1293 - (5) = 0.1293$ and $b = 0.2983 - (0) = 0.2983$

Therefore $[x_6] = 5 + 0 + 0 = 5 \Rightarrow a_6 = 5$.

$$x_7 = \frac{1}{[(5.1293 + 0.2983) - (5)]} \Rightarrow x_7 = 1.2233 - 2.8221i$$

Then $a = 1.2233 - (1) = 0.2233$ and $b = -2.8221 - (-3) = 0.4012$

Therefore $[x_7] = 1 - 3i + 0 = 1 - 3i \Rightarrow a_7 = 1 - 3i$.

$$x_8 = \frac{1}{[(1.2233 - 2.8221i) - (1 - 3i)]}$$

$$\Rightarrow x_8 = 2.7395 - 2.1825i$$

Then $a = 2.7395 - (2) = 0.7395$ and $b = -2.1825 - (-3) = 0.8125$

Therefore $[x_8] = 2 - 3i + i = 2 - 2i \Rightarrow a_8 = 2 - 2i$.

Continued fraction of the complex number $\alpha - i\beta = 1.1 - 0.9950\,i$

Take $x_0 = 1.1 - 0.9950i$

Then $a = 1.1 - (1) = 0.1000$ and $b = -0.9950 - (-1) = 0.0050$

Therefore $[x_0] = 1 - i + 0 = 1 - i \Rightarrow a_0 = 1 - i$.

Now

$$x_1 = \frac{1}{[(1.1 - 0.9950i) - (1 - i)]} \Rightarrow x_1 = 9.9751 - 0.4988i$$

Round off the imaginary part to 0.5, $x_1 = 9.9751 - 0.5i$

Then $a = 9.9751 - (9) = 0.9751$ and $b = -0.5 - (-1) = 0.5$

Therefore $[x_1] = 9 - i + 1 = 10 - i \Rightarrow a_1 = 10 - i$.

Again

$$x_2 = \frac{1}{[(9.9751 - 0.5i) - (10 - i)]} \Rightarrow x_2 = -0.0993 - 1.9951i$$

Then $a = -0.0993 - (-1) = 0.0007$ and $b = -1.9951 - (-2) = 0.0049$

Therefore $[x_2] = -1 - 2i + 0 = -1 - 2i \Rightarrow a_2 = -1 - 2i$.

$$x_3 = \frac{1}{[(-0.0993 - 1.9951i) - (-1 - 2i)]} \Rightarrow x_3 = 1.1102 - 0.0060i$$

Then $a = 1.1102 - (1) = 0.1102$ and $b = -0.0060 - (-1) = 0.9940$

Therefore $[x_3]=1-i+i=1 \Rightarrow a_3=1$.

$$x_4=\frac{1}{[(1.1102-0.0060i)-(1)]} \Rightarrow x_4=9.0476+0.4926i$$

Then $a=9.0476-(9)=0.0476$ and $b=0.4926-(0)=0.4926$

Therefore $[x_4]=9+0+0=7 \Rightarrow a_4=9$.

$$x_5=\frac{1}{[(9.0476+0.4926i)-(9)]}$$

$$\Rightarrow x_5=0.1943-2.0113i$$

Then $a=0.1943-(0)=0.1943$ and $b=-2.0113-(-3)=0.9887$

Therefore $[x_5]=0-3i+i=-2i \Rightarrow a_5=-2i$.

$$x_6=\frac{1}{[(0.1943-2.0113i)-(-2i)]}$$

$$\Rightarrow x_6=5.1293+0.2983i$$

Then $a=5.1293-(5)=0.1293$ and $b=0.2983-(0)=0.2983$

Therefore $[x_6]=5+0+0=5 \Rightarrow a_6=5$.

$$x_7=\frac{1}{[(5.1293+0.2983)-(5)]} \Rightarrow x_7=1.2233-2.8221i$$

Then $a=1.2233-(1)=0.2233$ and $b=-2.8221-(-3)=0.4012$

Therefore $[x_7]=1-3i+0=1-3i \Rightarrow a_7=1-3i$.

$$x_8=\frac{1}{[(1.2233-2.8221i)-(1-3i)]}$$

$$\Rightarrow x_8=2.7395-2.1825i$$

Then $a=2.7395-(2)=0.7395$ and $b=-2.1825-(-3)=0.8125$

Therefore $[x_8]=2-3i+i=2-2i \Rightarrow a_8=2-2i$.

Therefore the continued fraction of the complex number and its conjugate is

$$1.1+0.9950\,i=\langle 1+i, 10, -1-2i, 1, 9, -2i, 5, 1-3i, 2-2i, \cdots\rangle$$

and

$$1.1-0.9950\,i=\langle 1-i, 10-i, -1-2i, 1, 9, -2i, 5, 1-3i, 2-2i, \cdots\rangle$$

Using theorem we can identify the complex number of the above continued fraction

Consider the continued fraction

$$[1+i,\ 10,\ -1-2i,\ 1,\ 9,\ -2i,\ 5,\ 1-3i,\ 2-2i\cdots]$$

Here

$\alpha_0=1,\ \alpha_1=10,\ \alpha_2=-1,\ \alpha_3=1,\ \alpha_4=9,\ \alpha_5=0,\ldots$

$\beta_0=1,\ \beta_1=0,\ \beta_2=-2,\ \beta_3=0,\ \beta_4=0, \beta_5=-2,\ldots$

Then $\dfrac{p_0}{q_0}=1+i$

$$\frac{p_1}{q_1}=\frac{11+i}{10}=1.1+0.1i$$

$$\frac{p_2}{q_2}=\frac{-9-23i}{-9-20i}=1.1247+0.0561i$$

$$\frac{p_3}{q_3}=\frac{2-22i}{1-20i}=1.1022+0.0448i$$

$$\frac{p_4}{q_4}=\frac{9-221i}{-200i}=1.105+0.045i$$

$$\frac{p_5}{q_5}=\frac{-440-40i}{-399}=1.1+0.1003i$$

which is approximately equal to the complex number $1.1+0.9950i$

Next to consider the continued fraction

$$[1-i,\ 10-i, -1-2i, 1, 9, -2i,\ 5, 1-3i, 2-2i\cdots]$$

Here

$\alpha_0=1, \alpha_1=10, \alpha_2=-1, \alpha_3=1, \alpha_4=9, \alpha_5=0,\ldots$

$\beta_0=-1, \beta_1=-1, \beta_2=-2, \beta_3=0,\ \beta_4=0, \beta_5=-2,$

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Then $\frac{p_0}{q_0} = 1 - i$

$$\frac{p_1}{q_1} = \frac{10 - 11i}{10 - i} = 1.0990 - 0.9900i$$

$$\frac{p_2}{q_2} = \frac{-31 - 10i}{-11 - 19i} = 1.1016 - 0.9938i$$

$$\frac{p_3}{q_3} = \frac{-21 - 21i}{-1 - 20i} = 1.0998 - 0.9950i$$

$$\frac{p_4}{q_4} = \frac{-220 - 199i}{-20 - 199i} = 1.0999 - 0.9950i$$

$$\frac{p_5}{q_5} = \frac{-419 + 419i}{-399 + 20i} = 1.0999 - 0.9950i$$

which is approximately equal to the complex conjugate $1.1 - 0.9950i$

III. CONCLUSION

Continued fraction method of solving a quadratic equation is a lengthy procedure. In this method the roots may have infinite or periodic representation. Also occurrences of complex roots for a quadratic equation is common. It is well known that complex roots occur in pairs. In this paper, an attempt has been made to represent a complex number and its conjugate in terms of continued fractions. In future, we transform that into finding complex roots of a quadratic equation.

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AUTHORS PROFILE

A.Gnanam, Assistant professor ,Department of Mathematics, Government Arts College, Trichy-22. Email: gnaanam@yahoo.com

S.Krithika, Assistant professor ,Department of Mathematics, Seethalakshmi Ramaswami College, Trichy-2
Email:krithikasrc@gmail.com