Solving Singularly Perturbed Differential-Difference Equations with Dual Layer Using Fourth Order Numerical Method

V. Vidyasagar, MadhuLatha K, B. Ravindra Reddy

Abstract: ***In this paper, we presented a fourth-order numerical method to solve SPDDE with the dual-layer. The answer to the problem shows dual-layer behavior. A fourth-order finite difference plan on a uniform mesh is developed. The result of the delay and also advance parameters on the boundary layer(s) has likewise been evaluated as well as represented in charts. The applicability of the planned plan is actually confirmed through executing it on model examples. To show the accuracy of the method, the results are presented in terms of maximum absolute errors.***

Keywords: ***differential-difference equations, central differences, fourth order, dual layer.***

I. INTRODUCTION

Mathematically, any ordinary differential equation in which the highest derivative is multiplied by a small positive parameter and also have at least one shift term(delay or advance) is called singularly perturbed differential-difference equation(SPDDE). Issues requiring these formulas develop in investigation researches of control theory[1], in identifying the anticipated possibility for the age of activity options in afferent neuron close to arbitrary synaptic inputs in dendrites [2], in the modelling of account activation of a nerve cell [3]. Throughout one of the most present number of many years lots of evaluation work has actually been accomplished in acquiring the numerical options to these differential-difference solutions, to show a couple of are Lange as well as Miura [4-6], Kadalbajoo and also Sharma[7-10], Nageshwar Rao and Pramod Chakravarthy[11], Duressa *et al.*[12] and many more.

II. DESCRIPTION OF THE METHOD

Think about SPDDE along with little delay and additionally advance parameters of the kind:

$$\varepsilon^2 u''(x) + b(x)u(x-\delta) + c(x)u(x) + d(x)u(x+\eta) = f(x) \quad (1)$$

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$\forall x \in (0,1)$ and under the interval and boundary conditions

$$u(x) = \varphi(x), \text{ on } -\delta \le x \le 0 \quad (2)$$

$$u(x) = \gamma(x), \text{ on } 1 \le x \le 1+\eta \quad (3)$$

When $b(x), c(x), d(x), f(x), \varphi(x)$ and $\gamma(x)$ are sufficiently smooth functions on (0,1), the singular perturbation parameter ε is small positive parameter ($0 < \varepsilon << 1$), and $0 < \delta = o(\varepsilon)$ and $0 < \eta = o(\varepsilon)$ are the delay (negative shift) and the advance (positive shift) parameters respectively.

By using Taylor series expansion in the neighbourhood of the point *x*, we have

$$u(x-\delta) \approx u(x) - \delta u'(x) \quad (4)$$

$$u(x+\eta) \approx u(x) + \eta u'(x) \quad (5)$$

Using Eq. (4) and Eq.(5) in Eq.(1) we get an asymptotically equivalent singularly perturbed boundary value problem of the form:

$$\varepsilon^2 u''(x) + \alpha(x)u'(x) + \beta(x)u(x) = f(x) \quad (6)$$

$$u(0) = \varphi(0) = \varphi_0 \quad (7)$$

$$u(1) = \gamma(1) = \gamma_1 \quad (8)$$

where $\alpha(x) = d(x)\eta - b(x)\delta$

and $\beta(x) = b(x) + c(x) + d(x)$

Since $0 < \delta << 1$ and $0 < \eta << 1$,

The transition from Eq.(1) to Eq.(6) is admissible. Further details on the validity of this transition is found in El'sgol'ts and Norkin[13]. If $\beta \le 0$ on the interval [0,1], after that the solution of Eq.(1) exhibits boundary layers at each edges of the interval [0,1], whereas it exhibits oscillatory behaviour for $\beta(x) > 0$. Now, we considered dual layer problems.On discretizing the interval [0,1] in to N identical subintervals of mesh measurement h = 1/N to make sure that

$$x_i = x_0 + ih : i = 0, 1, 2, \ldots, N.$$

Let $u_i = u(x_i)$ for $x_i \in [0,1]$.

Assuming that *u(x)* has continuous derivatives on [0,1] and making use of Taylor's series expansions of u_{i+1} and u_{i-1} upto $O(h^7)$, we get the finite difference approximations for u_i' and u_i'' as

$$u_i' = \frac{u_{i+1} - u_{i-1}}{2h} - \frac{h^2}{6}u_i''' - \frac{h^4}{120}u_i^{(5)} + \tau_1 \quad (9)$$

where, $\tau_1 = -\frac{h^6}{7!}u^{(7)}(\xi_1)$, for $\xi_1 \in [x_{i-1}, x_i]$

$$u_i'' = \frac{u_{i+1} - 2u_i - u_{i-1}}{h^2} - \frac{h^2}{12}u_i^{(4)} - \frac{h^4}{360}u_i^{(6)} + \tau_2 \qquad (10)$$

where, $\tau_2 = -\frac{h^8}{8!}u^{(8)}(\xi_2)$, for $\xi_2 \in [x_{i-1}, x_i]$

Substituting Eqs. (9) and (10) into Eq. (6) and simplifying, we obtain:

$$\frac{1}{h^2}(u_{i+1} - 2u_i - u_{i-1}) + \frac{\alpha_i}{2h}(u_{i+1} - u_{i-1}) - \frac{h^2}{6}\alpha_i u_i''' - \frac{h^2}{12}u_i^{(4)} - \frac{h^4}{120}\alpha_i u_i^{(5)} + \beta_i y_i = r_i + \tau \qquad (11)$$

where, $\tau = \frac{h^4}{360}u^{(6)}(\xi_2) - \alpha_i \tau_i - \tau_2$ is the local truncation error and

$$\alpha(x_i) = \alpha_i, \ \beta(x_i) = \beta_i, r(x_i) = r_i$$

By successively differentiating both sides of Eq. (6) and evaluating at x_i, and using into Eq.(11), we obtain:

$$\frac{1}{h^2}(u_{i+1} - 2u_i - u_{i-1}) + \frac{\alpha_i}{2h}(u_{i+1} - u_{i-1}) + P_i u_i'' + Q_i u_i' + R_i u_i = S_i, \text{ for } i = 1,2,---N-1 \qquad (12)$$

where

$$P_i = \frac{h^2}{6}\alpha_i^2 - \frac{h^2}{12}(\alpha_i^2 - 2\alpha_i' - \beta_i) - \frac{h^4}{120}\alpha_i(2\alpha_i\alpha_i' - 3\alpha_i'' - 3\beta_i' + \alpha_i(\alpha_i' + \beta_i) - \alpha_i(\alpha_i^2 - 2\alpha_i' - \beta_i))$$

$$Q_i = \frac{h^2}{6}\alpha_i(\alpha_i' + \beta_i) - \frac{h^2}{12}\left[\alpha_i(\alpha_i' + \beta_i) - \alpha_i'' - 2\beta_i'\right]$$

$$-\frac{h^4}{120}\alpha_i\left[\alpha_i'(\alpha_i' + \beta_i) + \alpha_i(\alpha_i'' + \beta_i') - \alpha_i''' - 3\beta_i'' + \alpha_i\beta_i' - (\alpha_i' + \beta_i)(\alpha_i^2 - 2\alpha_i' - \beta_i)\right]$$

$$R_i = \frac{h^2}{6}\alpha_i\beta_i' - \frac{h^2}{12}(\alpha_i\beta_i' - \beta_i'') - \frac{h^4}{120}\alpha_i\left[\alpha_i'\beta_i' + \alpha_i\beta_i'' - \beta_i''' - \beta_i'(\alpha_i^2 - 2\alpha_i' - \beta_i)\right] + \beta_i$$

$$S_i = r_i + \left[\frac{h^2}{12}\alpha_i + \frac{h^4}{120}\alpha_i(\alpha_i^2 - 3\alpha_i' - \beta_i)\right]r_i' + \left[\frac{h^2}{12} - \frac{h^4}{120}\alpha_i^2\right]r_i'' + \frac{h^4}{120}\alpha_i r_i'''$$

Now, using central difference approximation for u_i'' and u_i' in Eq(12) and further simplifying, we get:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \text{ for } i = 1,2,----N-1$$

where

$$E_i = \frac{1}{h^2} - \frac{\alpha_i}{2h} + \frac{P_i}{h^2} - \frac{Q_i}{2h}$$

$$F_i = \frac{2}{h^2} + \frac{2P_i}{h^2} - R_i$$

$$G_i = \frac{1}{h^2} + \frac{\alpha_i}{2h} + \frac{P_i}{h^2} + \frac{Q_i}{2h}$$

III. NUMERICAL EXAMPLES

The exact solution of singularly perturbed differential-difference equation:

$$\varepsilon^2 u''(x) + b(x)u(x-\delta) + c(x)u(x) + d(x)u(x+\eta) = f(x)$$

$\forall x \in (0,1)$ and under the interval and boundary conditions

$$u(x) = \varphi(x), \text{ on } -\delta \le x \le 0$$

$$u(x) = \gamma(x), \text{ on } 1 \le x \le 1+\eta$$

with constant coefficients(i.e, $b(x) = b, \ c(x) = c, d(x) = d, \ f(x) = f, \varphi(x) = \varphi \text{ and } \gamma(x) = \gamma$) is given by

$$u(x) = \frac{[(1-b-c-d)e^{m_2} - 1]e^{m_1 x} - [(1-b-c-d)e^{m_1} - 1]e^{m_2 x}}{(b+c+d)(e^{m_1} - e^{m_2})} + \frac{1}{b+c+d}$$

where

$$m_1 = \frac{(b\delta - d\eta) + \sqrt{(d\eta - b\delta)^2 - 4\varepsilon^2(b+c+d)}}{2\varepsilon^2},$$

$$m_2 = \frac{(b\delta - d\eta) - \sqrt{(d\eta - b\delta)^2 - 4\varepsilon^2(b+c+d)}}{2\varepsilon^2}$$

Example 1. Consider the problem with constant coefficients

$$\varepsilon^2 u''(x) - 2u(x-\delta) - u(x) - 2u(x+\eta) = 1,$$

$$\varphi(x) = 1, \ \gamma(x) = 0$$

The numerical results are presented in Tables I and II & Fig. 1 and 2.

Example 2. Consider the problem with constant coefficients

$$\varepsilon^2 u''(x) + 0.25u(x-\delta) - u(x) + 0.25u(x+\eta) = 1,$$

$$\varphi(x) = 1, \ \gamma(x) = 0$$

The numerical results are presented in Tables III and IV & Fig. 3 and 4.

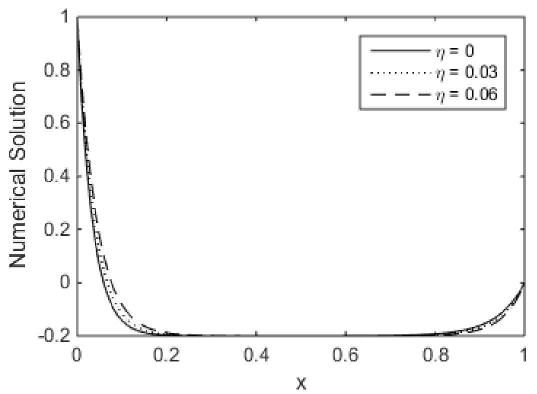

Fig 1. In Example 1, numerical solution for N=100, $\varepsilon = 0.1$ and $\delta = 0.07$

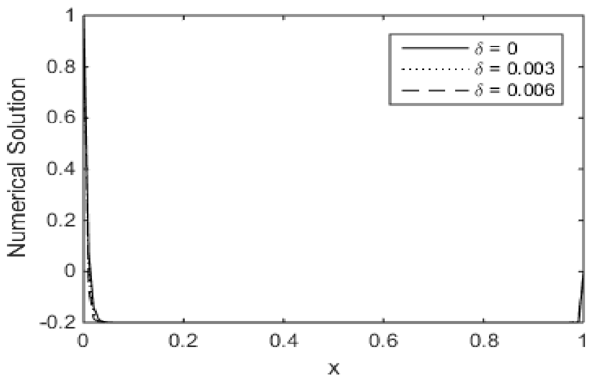

Fig 2. In Example 1, numerical solution for N=100, $\varepsilon = 0.01$ and $\eta = 0.007$

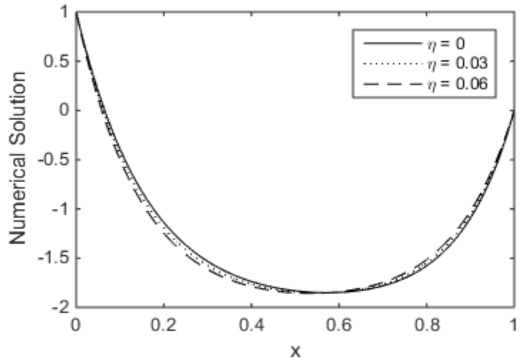

Fig 3. In Example 2, numerical solution for N=100, $\varepsilon = 0.1$ and $\delta = 0.07$

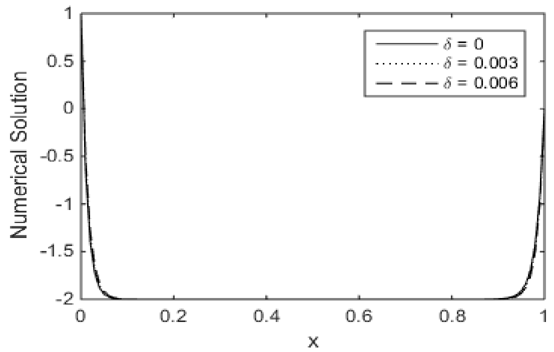

Fig 4. In Example 2, Numerical Solution for N=100, $\varepsilon = 0.01$ and $\eta = 0.007$

TABLE-I: In solution of Example 1, the numerical results for N=100, $\varepsilon = 0.1$ and $\delta = 0.07$

X	$\eta = 0$		$\eta = 0.03$		$\eta = 0.06$	
	Num. Solution	Exact Solution	Num. Solution	Exact Solution	Num. Solution	Exact Solution
0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.01	0.68516679	0.68516083	0.71866513	0.71866406	0.74979451	0.74979535
0.03	0.28162925	0.28161951	0.33840516	0.33840326	0.39501303	0.39501461
0.05	0.06205993	0.06205109	0.11554492	0.11554307	0.17275484	0.17275649
0.07	-0.05741021	-0.05741694	-0.01506753	-0.01506905	0.03351787	0.03351931
0.09	-0.12241525	-0.12241996	-0.09161600	-0.09161715	-0.05370923	-0.05370806
0.20	-0.19727026	-0.19727063	-0.19426272	-0.19426286	-0.18882752	-0.18882732
0.40	-0.19998333	-0.19998333	-0.19996991	-0.19996992	-0.19989544	-0.19989544
0.60	-0.19972028	-0.19972027	-0.19988773	-0.19988772	-0.19996045	-0.19996045
0.80	-0.19252066	-0.19252049	-0.19526414	-0.19526401	-0.19722212	-0.19722205
0.90	-0.16132354	-0.16132310	-0.16922384	-0.16922342	-0.17642937	-0.17642906
0.92	-0.14627661	-0.14627613	-0.15525159	-0.15525111	-0.16385045	-0.16385007
0.94	-0.12537574	-0.12537523	-0.13493601	-0.13493548	-0.14455854	-0.14455810
0.96	-0.09634347	-0.09634300	-0.10539724	-0.10539673	-0.11497112	-0.11497067
0.98	-0.05601630	-0.05601597	-0.06244801	-0.06244763	-0.06959381	-0.06959346
1.00	0.00000000	0.00000000	0.00000000	-0.00000000	0.00000000	0.00000000
Maximum Error : 9.7407e-06			1.9340e-06		1.6689e-06	

TABLE-II: In solution of Example 1, the maximum absolute errors for $\delta = \varepsilon^2$ and $\eta = 2\varepsilon^2$

$\varepsilon \backslash N$	128	256	512	1024	2048
0.1	1.0135e-06	6.3380e-08	3.9618e-09	2.4764e-10	1.5472e-11
0.01	6.8083e-03	5.2875e-04	3.3642e-05	2.1256e-06	1.3347e-07

TABLE-III: In solution of Example 2 , the numerical results for N=100, $\varepsilon = 0.01$ and $\eta = 0.007$

x	$\delta = 0$		$\delta = 0.003$		$\delta = 0.006$	
	Num. Solution	Exact Solution	Num. Solution	Exact Solution	Num. Solution	Exact Solution
0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.02	-1.39426880	-1.39431045	-1.34263834	-1.34238288	-1.28928582	-1.28881467
0.04	-1.87769657	-1.87771339	-1.85595855	-1.85584657	-1.83162845	-1.83140514
0.06	-1.97530567	-1.97531076	-1.96843756	-1.96840075	-1.96011198	-1.96003260
0.08	-1.99501396	-1.99501533	-1.99308402	-1.99307326	-1.99055034	-1.99052526
0.10	-1.99899327	-1.99899361	-1.99848457	-1.99848162	-1.99776133	-1.99775390
0.20	-1.99999966	-1.99999966	-1.99999923	-1.99999923	-1.99999833	-1.99999832
0.40	-2.00000000	-2.00000000	-2.00000000	-2.00000000	-2.00000000	-2.00000000
0.60	-2.00000000	-2.00000000	-2.00000000	-2.00000000	-2.00000000	-2.00000000
0.80	-1.99999261	-1.99999255	-1.99999625	-1.99999621	-1.99999817	-1.99999815
0.90	-1.99615601	-1.99613909	-1.99726018	-1.99724841	-1.99808471	-1.99807730
0.92	-1.98657136	-1.98652411	-1.98975804	-1.98972284	-1.99230880	-1.99228502
0.94	-1.95308826	-1.95296451	-1.96171356	-1.96161494	-1.96911461	-1.96904301
0.96	-1.83611809	-1.83583000	-1.85687792	-1.85663224	-1.87597420	-1.87578258
0.98	-1.42749338	-1.42699041	-1.46498209	-1.46452309	-1.50195222	-1.50156761
1.00	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
Maximum Error : 5.0298e-04			4.5900e-04		4.8391e-04	

TABLE-IV : In solution of Example 2, the maximum absolute errors for $\delta = \varepsilon^2$ and $\eta = 2\varepsilon^2$

$\varepsilon \backslash N$	128	256	512	1024	2018
0.1	1.9894e-08	1.2438e-09	7.8185e-11	4.8272e-12	5.7998e-13
0.01	2.0863e-04	1.3153e-05	8.2795e-07	5.1777e-08	3.2382e-09

IV. CONCLUSIONS

We examined fourth-order numerical method to solve singularly perturbed differential-difference equations showing a dual layer. To go over the applicability of the approach, we have resolved model instances through taking different values of N, ε, δ and η . The numerical solution is compared with the exact solution to examine the proposed method. We have presented maximum absolute errors for the conventional instances selected from the literary works. From the tables, the outcomes show that proposed method generated a good estimation to the exact solution. To analyze the impact of the criteria on the solution, the numerical outcomes have been outlined making use of graphs. From the figures, it is actually monitored that through varying the value of either δ or even η, the thickness of the left boundary layer lowers and that of the right boundary layer rises. The outcome of the negative shift is a whole lot much more leading than the positive shift.

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