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Finding Inverse of a Fuzzy Matrix using Eigen value Method

Hamed Farahani, M. J. Ebadi, Hossein Jafari

Abstract*: *The present paper extends a concept of the inverse of a matrix that its elements are fuzzy numbers, which may be implemented to model imprecise and uncertain features of the problems in the real world. The problem of inverse calculation of a fuzzy matrix is converted to solving a fuzzy polynomial equations (FPEs) system. In this approach, the fuzzy system is transformed to an equivalent system of crisp polynomial equations. The solutions of the crisp polynomial equations system is computed using eigenvalue method. Also, using Gröbner basis properties a criteria for invertibility of the fuzzy matrix is introduced. Furthermore, a novel algorithm is proposed to find a fuzzy inverse matrix. Achieving all entries of a fuzzy inverse matrix at a time is a big advantage comparing the existence methods. In the end, some illustrative examples are presented to demonstrate the algorithm and concepts.

Keywords* : *Eigenvalue, Fuzzy numbers, Fuzzy matrix, Fuzzy identity matrix, Fuzzy linear equation system.

I. INTRODUCTION

When the fuzzy uncertainty happens in a problem, the fuzzy matrices are successfully applied. In the last two decades, fuzzy matrices have been popular [24]. In matrix theory, the position of the theory of generalized inverse of a fuzzy matrix is outstanding [5, 6]. The research of convergence of powers of a fuzzy matrix began by Thomasan [22] in 1977. A systematic improvement to the fuzzy matrix theory was given by Kim and Roush [20]. Also, they proposed the algorithms to obtain a fuzzy inverse matrix and its generalized inverse. The 'fuzzy matrix' term is the principal idea of the present paper and has more than two various meanings in the research. In the first class A=(a_ij)_(m×n) is said to be a fuzzy matrix, if a_ij∈[0,1],(i=1,2,...,m;j=1,2,...,n). They have been first defined in detail and appeared with the fuzzy relations in [20]. Then, there was more attention in this case [7, 18, 21]. For example, the Gödel-implication operator was used by Hashimoto [18] and he presented some features of sub-inverse of the fuzzy matrices of the first kind. Also, the properties of their regularity were introduced by Cho in 1999 [7]. Moreover, a matrix including entries of fuzzy numbers is known as the fuzzy matrix, too [4, 11, 12, 19].

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* Correspondence Author
Hamed Farahani, Department of Mathematics, Chabahar Maritime University, Chabahar, Iran.
M. J. Ebadi*, Department of Mathematics, Chabahar Maritime University, Chabahar, Iran.
Hossein Jafari, Department of Mathematics, Chabahar Maritime University, Chabahar, Iran.

The study of the second class is ignored because of the complex arithmetic structure. In the present paper, the focus will be on this fuzzy matrices class. The investigations of the invertibility of the square interval matrices and obtaining their inverse are two popular problems that have been warm issues in recent studies. This paper proposes an approach to compute the fuzzy inverse matrix on the base of eigenvalue approach. In this method, finding the fuzzy inverse matrix is on the base of changing the fuzzy matrix into a crisp polynomial equations system. Then, a Gröbner basis regarding to a term order may be computed for the ideal which is generated by the crisp polynomials system. The Gröbner basis regarding each arbitrary term order can be computed. Moreover, the Gröbner basis computation w.r.t the lexicographical term order regarding the other terms of the order is more complex in the computational complexity viewpoint [1]. Hence, a suitable term order can be selected for computing the Gröbner basis and reducing the computations rate. In the eigenvalue approach, the roots calculation of a system is accomplished separately from each other. Accordingly, the occurred approximation and the probable error in the previous roots calculation don't influence the next roots computation. In the presented approach, the inverse computation of a fuzzy matrix is converted to obtaining the eigenvalues of a matrix. Therefore, the valuable tools from linear algebra can be used for instance to transform a matrix into triangular one through using the properties of determinant and elementary row operations. Also, a criteria is presented on the base of Gröbner basis for invertibility of the fuzzy matrix. The organization of the paper is as follows. In Section 2, some necessary results and definitions of fuzzy numbers are mentioned. Then, the necessary results and concepts of Gröbner basis and polynomials are given in Section 3. In addition, a new method for calculating the fuzzy inverse matrix is presented in Section 3. Also, a criteria and an algorithm are presented to find the fuzzy inverse matrix when it has inverse. Section 4 containing some examples which illustrate the algorithm. Our conclusions are summarized in Section 5.

II. FUZZY BACKGROUND

In this section some preliminaries and required background on fuzzy arithmetic, fuzzy numbers, fuzzy matrices and notation of fuzzy set theory are given.

Definition 2.1 *[10, 23] A fuzzy number M on the set of real numbers $\mathbb{R}$ is a fuzzy set if its membership function $\mu_M(x)$, $\mu_M:\mathbb{R} \to [0,1]$ is as follows:*

$$\mu_M(x) = \begin{cases} 0 & x \le a, \\ f_A(x) & a \le x \le c, \\ 1 & c \le x \le d, \\ g_A(x) & d \le x \le b, \\ 0 & x \ge b. \end{cases}$$

Definition 2.2 *The fuzzy number M with the following function of membership $\mu_M(x)$:*

$$\mu_M(x) = \begin{cases} 1 & x \in [\underline{m}, \overline{m}], \\ L(\frac{\underline{m}-x}{\alpha}) & x \le \underline{m}, \\ R(\frac{x-\overline{m}}{\beta}) & x \ge \overline{m}, \end{cases}$$

is called a number of LR type where the non-increasing continuous functions L and R defined on $[0, \infty]$ decrease strictly to zero in those subintervals of $[0, \infty]$ in which they are satisfying the conditions $L(0) = R(0) = 1$ and positive. Also, the non-negative real numbers α and β are the spread parameters. Usually, L and R are known as the shape functions. In addition, the following is parametric form of fuzzy numbers of type LR [11]:

$M = (m, \alpha, \beta)$, where left and right spreads are respectively α and β. The symmetric fuzzy number M is a fuzzy number in which the spreads are $\alpha = \beta$ [11].

The operations of arithmetic were defined by Dubois and Prade [11] relying on the parametric representations of the fuzzy numbers of LR type. Here, multiplication and addition are given for the purposes of illustration. All formulas and more detailed descriptions can be found in [11]. The addition and multiplication for two positive fuzzy numbers of LR type $\widetilde{M} = (m, \alpha, \beta)$ and $\widetilde{N} = (n, \gamma, \delta)$ are given respectively as follows:

$$\widetilde{M} \oplus \widetilde{N} = (m, \alpha, \beta)_{LR} \oplus (n, \gamma, \delta)_{LR} = (m + n, \alpha + \gamma, \beta + \delta)_{LR},$$

and

$$\widetilde{M} \otimes \widetilde{N} = (m, \alpha, \beta)_{LR} \otimes (n, \gamma, \delta)_{LR} = (mn, m\gamma + n\alpha, m\delta + n\beta)_{LR}.$$

It is noticeable that the outcome fuzzy number is a kind of approximate results. In the below, we can find the scalar multiplication:

$$\lambda \otimes \widetilde{N} = \begin{cases} (\lambda n, \lambda\gamma, \lambda\delta)_{LR} & \lambda > 0, \\ (\lambda n, -\lambda\delta, -\lambda\gamma)_{LR} & \lambda < 0. \end{cases}$$

The definitions of being positive, negative, and zero of a fuzzy number are given below:

Definition 2.3 *When $[a_1, a_2]$ is a fuzzy number support, if $0 \le a_1 \le a_2$ the fuzzy number considered as positive. Thus, if $a_1 \le a_2 < 0$ the fuzzy number considered as negative. Lastly, if $a_1 \le 0 \le a_2$ the fuzzy number considered as zero.*

Fuzzy matrix was defined and introduced as a rectangular array of fuzzy numbers [10, 11]. Thus, the definition in formal form was defined as below [9]:

Definition 2.4 *If the elements of a matrix $\tilde{A} = (\tilde{a}_{ij})$ are fuzzy numbers we say that $\tilde{A}$ is a fuzzy matrix. In addition, consider $\tilde{A} = (\tilde{a}_{ij})$ and $\tilde{B} = (\tilde{b}_{ij})$ as two fuzzy matrices of orders $m \times n$ and $n \times p$, respectively. The $m \times p$ is the product order of two fuzzy matrices and their product is given as below:*

$$\tilde{A} \otimes \tilde{B} = \tilde{C} = (\tilde{c}_{ij}),$$

where $\tilde{c}_{ij} = \sum_{k=1,\dots n}^{\oplus} \tilde{a}_{ik} \otimes \tilde{b}_{kj}$, where the approximated multiplication denoted by $\oplus$.

In an analogous manner, a fuzzy matrix A including the spread parts of the left and right and the center just as a fuzzy number is given as the following form: $\tilde{A} = (A, L, R)$, in which A, L and R represent the center, left and right spread matrices respectively and all are crisp. Moreover, the sizes of them are the same [9]. The interested reader can refer to [24] for further basic and essential properties of the fuzzy matrices.

III. POLYNOMIALS AND GRÖBNER BASIS

This section contains the introduction of some basic concepts in relation to the Gröbner basis and polynomials. Consider $\mathbb{K}$ as a field and $x_1, \dots, x_n$ as n (algebraically independent) variables. Each power product $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is called a monomial where $\alpha_1, \dots, \alpha_n \in \mathbb{Z}_{\ge 0}$. Because of simplicity, we abbreviate such monomials by x^α where x is used for the sequence $x_1, \dots, x_n$ and $\alpha = (\alpha_1, \dots, \alpha_n)$. The set of all monomials can be sort over $\mathbb{K}$ via special kinds of total orderings which known as the orderings of monomial recalled as the below definition.

Definition 3.1 *The total ordering $\prec$ on the monomials set is named orderings of monomial if for each x^α, x^β and x^γ as monomials we have:*

- $x^\alpha \prec x^\beta \Rightarrow x^\gamma x^\alpha \prec x^\gamma x^\beta$,

and

- $\prec$ is well-ordering.

There are infinitely many monomial orderings, each one is convenient for a special type of problems. Between them, the graded and pure reverse lexicographic orderings represented by $\prec_{grevlex}$ and $\prec_{lex}$ pointed out as follows: Assume that $x_n \prec \cdots \prec x_1$. We say that

- $x^\alpha \prec_{lex} x^\beta$ whenever $\alpha_1 = \beta_1, \dots, \alpha_i = \beta_i$ and $\alpha_{i+1} < \beta_{i+1}$ for an integer $1 \le i < n$.

- $x^\alpha <_{grevlex} x^\beta$ if $\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$,

breaking ties when there exists an integer $1 \leq i < n$ such that

$\alpha_n = \beta_n, \ldots, \alpha_{n-i} = \beta_{n-i}$ and $\alpha_{n-i-1} > \beta_{n-i-1}$.

It is worth noting that the former has many theoretical importance while the latter speeds up the computations and carries fewer information out. Any polynomial on $x_1, \ldots, x_n$ over $\mathbb{K}$ can be written as $\mathbb{K}$ −linear combination of monomials. The polynomial ring on $x_1, \ldots, x_n$ over $\mathbb{K}$ represented by $\mathbb{K}[x_1, \ldots, x_n]$ or just by $\mathbb{K}[x]$ and is the set of all polynomials having the structure of a ring with common polynomial multiplication and addition. The leading monomial of f is the greatest nominal with respect to $\prec$ included in f and represented by $LM(f)$. The leading coefficient of f is the coefficient of $LM(f)$ and denoted by $LC(f)$.Moreover, $LM(F)$ is said to be $\{LM(f) | f \in F\}$ if F is a polynomials set, and $in(I)$ is said to be the initial ideal of I and the ideal generated by $LM(I)$ if I is an ideal. Now, we decide to mention the idea of Gröbner basis of polynomial ideal which carries lots of useful information out about the ideal.

Definition 3.2 *Consider $\prec$ as a monomial ordering and I as a polynomial ideal of $K[x]$. The finite set $G \subset I$ is said to be a Gröbner basis of I if for any nonzero polynomial $f \in I$, and for some $g \in G$, $LM(f)$ is divisible by $LM(g)$.*

Using the famous basis theorem of Hilbert (See [2]), it is shown that any polynomial ideal holds a Gröbner basis w.r.t any monomial ordering. There exist also some efficient algorithms to calculate Gröbner basis. The first and the most simplest one is the Buchberger algorithm which is devoted in the same time of introducing the Gröbner basis concept while the most efficient known algorithm is the Faugère's F_5 algorithm [15] and another signature-based algorithms such as G^2V [16] and GVW [17]. It is worth noting that Gröbner basis of an ideal is not necessarily unique. To have uniquity, we define the reduced Gröbner basis concept. We have the uniqueness of the reduced Gröbner basis of an ideal up to the monomial ordering as a significant reality.

Definition 3.3 *Consider G as a Gröbner basis for the ideal I w.r.t. $\prec$. Then G is so called a reduced Gröbner basis of I whenever each $g \in G$ is monic, which means that $LC(g) = 1$ and for each $h \in G \backslash \{g\}$ none of the appearing monomials in g is divisible by $LM(h)$.*

The help of Gröbner basis to solve a polynomial system is one of its most applications. Consider

$$\begin{cases} f_1 & = & 0 \\ & \vdots & \\ f_k & = & 0 \end{cases}$$

as a polynomial system and $I = \langle f_1, \ldots, f_k \rangle$ as the ideal generated by $f_1, \ldots, f_k$. The affine variety corresponding to the above polynomial system or to the ideal I is defined to be

$$V(I) = V(f_1, \ldots, f_k) = \{\alpha \in \overline{\mathbb{K}}^n | f_1(\alpha) = \cdots = f_k(\alpha) = 0\}$$

where $\overline{\mathbb{K}}$ is used to denote the algebraic closure of $\mathbb{K}$. Now, consider G as a Gröbner basis for I w.r.t an arbitrary monomial ordering. As an interesting fact, $I = \langle G \rangle$ which indicates that $V(I) = V(G)$. This is the key computational trick to solve a polynomial system. Let us continue by an example.

Example 3.4 *We are going to solve the following polynomial system:*

$$\begin{cases} x^2 - xyz + 1 & = & 0 \\ y^3 + z^2 - 1 & = & 0 \\ xy^2 + z^2 & = & 0 \end{cases}$$

By the nice properties of pure lexicographical ordering, the reduced Gröbner basis of the ideal $I = \langle x^2 - xyz + 1, y^3 + z^2 - 1, xy^2 + z^2 \rangle \subset \mathbb{Q}[x, y, z]$ has the form

$$G = \{g_1(z), x - g_2(z), y - g_3(z)\}$$

with respect to $z \prec_{lex} y \prec_{lex} x$, where

$$\begin{cases} g_1(z) = z^{15} - 3z^{14} + 5z^{12} - 3z^{10} - z^9 - z^8 \\ \qquad + 4z^6 - 6z^4 + 4z^2 - 1 \\ g_2(z) = 2z^{14} - 9z^{13} + 11z^{12} + 2z^{11} - 7z^{10} \\ \qquad - 3z^9 + 2z^8 - z^7 + 4z^6 + 7z^5 - 10z^4 \\ \qquad - 6z^3 + 11z^2 + 2z - 4 \\ g_3(z) = z^{13} - 3z^{12} + z^{11} + 2z^{10} + z^9 \\ \qquad - z^8 - 2z^6 + 2z^4 - z^3 - 3z^2 + 1 \end{cases}$$

This special form of Gröbner basis for this system allows us to find $V(G)$ by solving only one univariate polynomial $g_1(z)$ and putting the roots into the two last polynomials in G.

Theorem 3.5 *Suppose that G is a reduced Gröbner basis for I w.r.t any monomial ordering and I is an ideal in $\mathbb{K}[x]$. If $G = \{1\}$ then $V(I) = \emptyset$.*

The existence of univariate polynomials in a polynomial ideal depends on the dimension of the ideal. The concept of dimension of an ideal is recalled in the next definition.

Definition 3.6 *Consider u as a set of variables and $I \subset \mathbb{K}[x]$ as an ideal. The set of variables u is said an independent set w.r.t I, whenever $\mathbb{K}[u] \cap I = \{0\}$.*

The dimension of I is the maximal independent set cardinality w.r.t I. Furthermore, when the dimension of I is zero I called a zero dimensional ideal, and positive dimensional otherwise. Zero dimensional ideals have very nice properties which facilitate the computations. For instance, for an ideal I with zero dimension, the dimension of the vector space $\mathbb{K}[x]/I$ is finite and one can find its basis easily via reading the leading monomials of a Gröbnr basis. A basis for $\mathbb{K}[x]/I$ in which the set of all monomials in $\mathbb{K}[x]$ denoted by $\mathbb{M}$ and can be constructed by the set

$$B = \mathbb{M} \setminus in(I).$$

More precisely, the computation of a Gröbner basis G at first is enough to compute B, and carry those monomials out which are not divisible by $LM(g)$ for each $g \in G$. A new property of the ideals with zero dimension in described in the below theorem in which it is one of the essential theorems in the present paper. However, the following definition should be given.

Definition 3.7 *Suppose that I is a polynomial ideal with zero dimension and B is a basis for $\mathbb{K}[x]/I$. The definition of the linear transformation φ_h for every polynomial $h \in \mathbb{K}[x]$ is as below:*

$$\varphi_h : \frac{\mathbb{K}[x]}{I} \to \frac{\mathbb{K}[x]}{I}$$

$$f + I \mapsto hf + I$$

Also, suppose that M_h is the matrix representation of φ_h w.r.t B. Therefore, M_h is said the multiplication matrix of h w.r.t I.

IV. PROPOSED METHODOLOGY

4.1 Eigenvalues for solving 0-D polynomial system

The following theorem and algorithm state the method of using eigenvalues to solve a zero dimensional polynomial system.

Theorem 4.1 ([2]) *Using the above notations, the eigenvalues of M_h show the values of h over $V(I)$.*

As a very fast conclusion of this theorem, one can solve a zero dimensional polynomial equations system by calculating the eigenvalues of M_{x_i} for each variable x_i. Of course the eigenvalues of M_{x_i} are the i –th component of $V(I)$.

Algorithm 1. Eigenvalue Method

Require: $F = \{f_1, \ldots, f_k\} \subset \mathbb{K}[x]$; a set of polynomials where $x = x_1, \ldots, x_n$
Ensure: $V(F)$
$G :=$ a Gröbner basis for $\langle F \rangle$ w.r.t an arbitrary monomial ordering;
$B :=$ a basis for $\frac{\mathbb{K}[x]}{\langle F \rangle}$;
for $i = 1, \ldots, n$ **do**
$E_i :=$ the eigenvalue set of M_{x_i};
end for
$V := E_1 \times \cdots \times E_n$;
for $v \in V$ **do**
if $f_i(v) \neq 0$ for an $i = 1, \ldots, k$ **then**
$V := V \setminus \{v\}$;
end if
end for
Return (V);

Using linear algebra, eigenvalue method is a simple and efficient method to solve a zero dimensional ideal. However, the result of cartesian product of eigenvalues gives a superset of the solution set and so it is needed, as mentioned in the algorithm, to check whether a tuple is a solution or not. For instance to solve Example 3.4 by this method we need to check 15^3 tuples to find out whether they belong to the solution set or not. This is while this system has only 15 solutions. Thus, this method is convenient when the degree of univariate is low w.r.t the number of variables.Now, the eigenvalue approach illustrated to find the real solutions of a polynomial system through the below example.

Example 4.2 *[13] The following system of equations is considered:*

$$\begin{cases} x^3 + y^3 + z^3 - xyz = -4, \\ xy + xz + yz = -3, \\ x^2 + y^2 + z^2 = 6. \end{cases}$$

Suppose that I is the generated ideal via these equations.

At first, a Gröbner basis for I w.r.t the graded reverse lexicographic order is computed and so the command

$$|NormalSet(G, tdeg(x, y, z))|$$

is used to find the below monomials:

$$B = \{1, x, y, z, z^2, yz, xz, y^2, z^3, yz^2, y^2z, yz^3\}$$

Now, the matrix representation of φ_x w.r.t B is obtained as follows:

$$M_x = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 2 & \frac{3}{2} & \frac{9}{2} & \frac{9}{2} & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -2 & -\frac{9}{2} & -\frac{9}{2} & -\frac{9}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 2 & \frac{9}{2} & \frac{3}{2} & \frac{9}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & \frac{3}{2} & \frac{9}{2} & \frac{9}{2} & 0 & 0 & 0 & 0 & -1 \\ 0 & -2 & 0 & 0 & -\frac{9}{2} & -\frac{9}{2} & -\frac{9}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{27}{2} & 0 & -2 & 0 & 0 & 0 & \frac{9}{2} & -\frac{9}{2} & 0 & 0 & 0 & 0 \\ -9 & -\frac{27}{4} & -\frac{81}{4} & -\frac{81}{4} & -2 & 0 & 0 & 0 & -\frac{9}{2} & 0 & 0 & 0 \end{bmatrix}$$

The "MultiplicationMatrix" command in Maple can also be used to compute this matrix. Now, $\{1,-2\}$ as two real eigenvalues can be obtained by using "*Eigenvalues*" command in Maple. Similarly, the values y and z can be computed. The eigenvalues of M_y, M_z are respectively $\{1,-2\}$ and $\{1,-2\}$. In the end, below solutions are obtained.

$$V(I) = \{(1,1,-2),(1,-2,1),(-2,1,1)\}.$$

4.2 Calculating Inverse of Fuzzy Matrix

This section presents some definitions to help the calculation of fuzzy inverse matrix. Then, a new approach is presented to calculate the fuzzy inverse matrix on the base of Gröbner basis benefits.

Definition 4.3 *Suppose that ε is the value of left and right spreads where $0 < \varepsilon < 1$ and ε is sufficiently small and the fuzzy number center value is 1. This fuzzy number represented by $\tilde{1} = (1,\varepsilon)$ and known as fuzzy one number.*

It can be clearly seen that the values of the spread value ε, will be taken between 0 and 1. Thus, for fuzzy one number, end points changes of the left and right are between $0 < 1-\varepsilon$ and $1+\varepsilon < 2$. So, an analogous definition is given for zero fuzzy number as follows:

Definition 4.4 *Suppose that δ is the value of left and right spreads in which δ is sufficiently small and $0 < \delta < 1$. Also, the fuzzy number center value is 0. This fuzzy number denoted by $\tilde{0} = (0,\delta)$ and known as the fuzzy zero number.*

Thus, for fuzzy zero number, end points changes of the left and right are between $-1 < 0-\delta$ and $0+\delta < 1$. So, the following is the definition of the fuzzy identity matrix:

Definition 4.5 *A fuzzy identity matrix is a fuzzy matrix in which its diagonal and off-diagonal elements are respectively fuzzy one and zero numbers and is denoted by $\tilde{I}$ as follows.*

$$\tilde{I} = \begin{bmatrix} \tilde{1} & \vdots & \tilde{0} \\ \vdots & & \vdots \\ \tilde{0} & & \tilde{1} \end{bmatrix}$$

Now, a new method is presented for calculating the fuzzy inverse matrix using the Gröbner basis.

Definition 4.6 *Suppose $\tilde{A}$ and $\tilde{X}$ are LR type fuzzy matrices of size n which is given as follows:*

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & \vdots & \tilde{a}_{1n} \\ \vdots & & \vdots \\ \tilde{a}_{n1} & & \tilde{a}_{nn} \end{bmatrix} \qquad \tilde{X} = \begin{bmatrix} \tilde{x}_{11} & \vdots & \tilde{x}_{1n} \\ \vdots & & \vdots \\ \tilde{x}_{n1} & & \tilde{x}_{nn} \end{bmatrix}$$

where both a_{ij} and x_{ij}, $i,j = 1,2,\dots,n$ are triangular fuzzy numbers. Then $\tilde{A}$ is invertible and $\tilde{X}$ is the inverse of $\tilde{A}$, if

$$\tilde{A} \otimes \tilde{X} = \tilde{I}. \qquad (1)$$

The expression of equation (1) is rewritten as follows:

$$\begin{bmatrix} \tilde{a}_{11} & \vdots & \tilde{a}_{1n} \\ \vdots & & \vdots \\ \tilde{a}_{n1} & & \tilde{a}_{nn} \end{bmatrix} \otimes \begin{bmatrix} \tilde{x}_{11} & \vdots & \tilde{x}_{1n} \\ \vdots & & \vdots \\ \tilde{x}_{n1} & & \tilde{x}_{nn} \end{bmatrix} = \begin{bmatrix} \tilde{1} & \vdots & \tilde{0} \\ \vdots & & \vdots \\ \tilde{0} & & \tilde{1} \end{bmatrix},$$

where $\tilde{a}_{ij} = (a_{ij}, b_{ij}, c_{ij})$, $\tilde{x}_{ij} = (x_{ij}, w_{ij}, z_{ij})$, $i,j = 1,2,\dots,n$ and $\tilde{1} = (1,\varepsilon)$, $\tilde{0} = (0,\delta)$.

Then, we write the matrix multiplication in the system of fuzzy linear equations as below:

$$\begin{cases} (\tilde{a}_{i1} \otimes \tilde{x}_{1i}) \oplus (\tilde{a}_{i2} \otimes \tilde{x}_{2i}) \oplus \dots \oplus (\tilde{a}_{in} \otimes \tilde{x}_{ni}) = \tilde{1}, \; i = 1,2,\dots,n, \\ (\tilde{a}_{i1} \otimes \tilde{x}_{1j}) \oplus (\tilde{a}_{i2} \otimes \tilde{x}_{2j}) \oplus \dots \oplus (\tilde{a}_{in} \otimes \tilde{x}_{nj}) = \tilde{0}, i \neq j = 1,2,\dots,n. \end{cases} \qquad (2)$$

The system of equations in (2) is written as below:

$$\begin{cases} ((a_{i1}, b_{i1}, c_{i1}) \otimes (x_{1i}, w_{1i}, z_{1i})) \oplus \dots \oplus ((a_{in}, b_{in}, c_{in}) \otimes (x_{ni}, w_{ni}, z_{ni})) = (1,\varepsilon,\varepsilon) \\ ((a_{i1}, b_{i1}, c_{i1}) \otimes (x_{1j}, w_{1j}, z_{1j})) \oplus \dots \oplus ((a_{in}, b_{in}, c_{in}) \otimes (x_{nj}, w_{nj}, z_{nj})) = (0,\delta,\delta) \end{cases} \qquad (3)$$

where ε, δ are sufficiently small. Now, the system (3) is converted into the below crisp system:

$$\begin{cases} a_{i1}x_{1i} + a_{i2}x_{2i} + \dots + a_{in}x_{ni} = 1 & i = 1,2,\dots,n, \\ (a_{i1}w_{1i} + b_{i1}x_{1i}) + \dots + (a_{in}w_{ni} + b_{in}x_{ni}) = \varepsilon & i = 1,2,\dots,n, \\ (a_{i1}z_{1i} + c_{i1}x_{1i}) + \dots + (a_{in}z_{ni} + c_{in}x_{ni}) = \varepsilon & i = 1,2,\dots,n, \\ & \\ a_{i1}x_{1j} + a_{i2}x_{2j} + \dots + a_{in}x_{nj} = 0 & i \neq j = 1,2,\dots,n, \\ (a_{i1}w_{1j} + b_{i1}x_{1j}) + \dots + (a_{in}w_{nj} + b_{in}x_{nj}) = \delta & i \neq j = 1,2,\dots,n, \\ (a_{i1}z_{1j} + c_{i1}x_{1j}) + \dots + (a_{in}z_{nj} + c_{in}x_{nj}) = \delta & i \neq j = 1,2,\dots,n, \end{cases} \qquad (4)$$

which is a polynomial system with $3s$ equations and $3n$ unknowns. Now, a Gröbner basis is calculated for the ideal generated by system (4) in the ring $R = \mathbb{R}[x_{ji}, w_{ji}, z_{ji}]$ (for $i,j = 1,2,\dots,n$) w.r.t the graded lexicographic order. Then, by using eigenvalue method all of the solutions of system (4) can be obtained.

Theorem 4.7 *If G is a Gröbner basis for the ideal generated by system (4). Then, matrix $\tilde{A}$ is invertible if and only if $G \neq \{1\}$.*

Proof. Consider F as a set of polynomials in system (4). Thus, with attention to Theorem 3.5, $V(F)$ is nonemty if $G \neq \{1\}$, therefore $V(F)$ will be a solution of the system (2). The proof of the converse is easy and left to the reader. Now, based on the above discussions, the following algorithm presents the process of calculating fuzzy inverse matrix:

Algorithm 2. Main Algorithm

Require: LR type fuzzy matrix $\hat{A}$ of size n **Ensure:** Fuzzy inverse matrix of $\hat{A}$

[1] Compute the fuzzy linear equation system (2) that allows calculating the inverse of the matrix $\hat{A}$

[2] Compute the crisp form of system (2) as system (4)

[3] Compute G as a graded lexicographical Gröbner basis for the ideal generated by system (4).

[4] If $G = \{1\}$ then go to [6]

[5] Find the solutions of system (4) using eigenvalue method as entries of the fuzzy inverse matrix $\hat{A}$

[6] LR type fuzzy matrix $\hat{A}$ is not invertible

[7] **End**

V. RESULT ANALYSIS

Some numerical examples are given in this section to illustrate the efficiency of the proposed method. Note that for the sake of simplicity all used fuzzy numbers in the following are symmetric triangular fuzzy numbers.

Example 5.1 *Consider the following fuzzy matrix of order 2×2 including the symmetric triangular fuzzy number elements:*

$$\tilde{A} = \begin{bmatrix} (10,4) & (8,3) \\ (6,2) & (4,3) \end{bmatrix}.$$

For calculating the fuzzy inverse matrix $\tilde{A}$, we construct its real case and extend to the fuzzy case as below:

$$\begin{bmatrix} (10,4) & (8,3) \\ (6,2) & (4,3) \end{bmatrix} \otimes \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{bmatrix} = \begin{bmatrix} \tilde{1} & \tilde{0} \\ \tilde{0} & \tilde{1} \end{bmatrix}.$$

where all unknowns $\tilde{x}_{ij}$ are represented by (x_{ij}, w_{ij}) for $i, j = 1,2$. Then, we can write the matrix multiplication in the fuzzy linear equation system form as below:

$$\begin{cases} ((10,4) \otimes \tilde{x}_{11}) \oplus ((8,3) \otimes \tilde{x}_{21}) = (1, \varepsilon), \\ ((10,4) \otimes \tilde{x}_{12}) \oplus ((8,3) \otimes \tilde{x}_{22}) = (0, \delta), \\ ((6,2) \otimes \tilde{x}_{11}) \oplus ((4,3) \otimes \tilde{x}_{21}) = (0, \delta), \\ ((6,2) \otimes \tilde{x}_{12}) \oplus ((4,3) \otimes \tilde{x}_{22}) = (1, \varepsilon). \end{cases}$$

For the given example, we can write the above equation system as follows:

$$\begin{cases} ((10,4) \otimes (x_{11}, w_{11})) \oplus ((8,3) \otimes (x_{21}, w_{21})) = (1, \varepsilon), \\ ((10,4) \otimes (x_{12}, w_{12})) \oplus ((8,3) \otimes (x_{22}, w_{22})) = (0, \delta), \\ ((6,2) \otimes (x_{11}, w_{11})) \oplus ((4,3) \otimes (x_{21}, w_{21})) = (0, \delta), \\ ((6,2) \otimes (x_{12}, w_{12})) \oplus ((4,3) \otimes (x_{22}, w_{22})) = (1, \varepsilon). \end{cases}$$

The equivalence of the above equation system and the following system is found by implementing the approximate fuzzy multiplication:

ε	w_{11}	w_{21}	w_{1}	w_{22}
1×10^{-1}	-1.075	1.325	1.925	-2.425
1×10^{-2}	-1.120	1.370	1.880	-2.380
1×10^{-3}	-1.1245	1.3745	1.8755	-2.3755
1×10^{-4}	-1.12495	1.37495	1.87505	-2.37505
1×10^{-5}	-1.124995	1.374995	1.875005	-2.375005
1×10^{-6}	-1.1249995	1.3749995	1.8750005	-2.3750005
1×10^{-7}	-1.12499995	1.37499995	1.87500005	-2.37500005
1×10^{-8}	-1.124999995	1.374999996	1.875000005	-2.375000004

$$\begin{cases} 10x_{11} + 8x_{21} = 1, \\ 10x_{12} + 8x_{22} = 0, \\ 6x_{11} + 4x_{21} = 0, \\ 6x_{12} + 4x_{22} = 1, \\ 10w_{11} + 4x_{11} + 8w_{21} + 3x_{21} = \varepsilon, \\ 10w_{12} + 4x_{12} + 8w_{22} + 3x_{22} = \delta, \\ 6w_{11} + 2x_{11} + 4w_{21} + 3x_{21} = \delta, \\ 6w_{12} + 2x_{12} + 4w_{22} + 3x_{22} = \varepsilon. \end{cases}$$

The Gröbner basis for the ideal generated by the above system w.r.t the reverse graded lexicographic order in the ring $R = \mathbb{R}[x_{11}, x_{21}, x_{12}, x_{22}, w_{11}, w_{21}, w_{12}, w_{22}]$ is as:

$\{19 - 6\delta + 10\varepsilon + 8w_{22}, -15 + 4\delta - 8\varepsilon + 8w_{12}, -11 - 6\varepsilon + 10\delta + 8w_{21}, 9 + 4\varepsilon - 8\delta + 8w_{11}, 5 + 4x_{22}, x_{12} - 1, -3 + 4x_{21}, 2x_{11} + 1\}$.

So we obtain the monomial basis $B = \{1\}$. The eigenvalues of the matrices of the full multiplication operator $M_{x_{11}}, M_{x_{21}}, M_{x_{12}}, M_{x_{22}}, M_{w_{11}}, M_{w_{21}}, M_{w_{12}}$ and $M_{w_{22}}$ can be obtained as $-0.5, 0.75, 1, -1.25, -1.125 - 0.5\varepsilon + \delta, 1.375 + 0.75\varepsilon - 1.25\delta, 1.875 - 0.5\delta + \varepsilon$ and $-2.375 + 0.75\delta - 1.25\varepsilon$, respectively. Therefore, the solution of the system is as

$x_{11} = -0.5, x_{21} = 0.75, x_{12} = 1, x_{22} = -1.25, w_{11} = -1.125 - 0.5\varepsilon + \delta, w_{21} = 1.375 + 0.75\varepsilon - 1.25\delta, w_{12} = 1.875 - 0.5\delta + \varepsilon, w_{22} = -2.375 + 0.75\delta - 1.25\varepsilon$

Note that the values of w_{ij} are found depending upon α, δ parameters unless those specified in advance. In addition, note that because of the LR type fuzzy number definition, the spread values of fuzzy numbers of LR type cannot be negative. When the values of spread are negative we should take their absolute values. For example, if $\alpha = \delta = 1 \times 10^{-5}$ then we can write the inverse of $\tilde{A}$ as follows:

$$\begin{bmatrix} (-0.5, |-1.124995|) & (1, 1.875005) \\ (0.75, 1.374995) & (-1.25, |-2.375005|) \end{bmatrix}$$

$$= \begin{bmatrix} (-0.5,1.124995) & (1,1.875005) \\ (0.75,1.374995) & (-1.25,2.375005) \end{bmatrix}$$

Therefore, we have

$$A.\tilde{A}^{-1} =$$
$$\begin{bmatrix} (10,4) & (8,3) \\ (6,2) & (4,3) \end{bmatrix} . \begin{bmatrix} (-0.5,1.124995) & (1,1.875005) \\ (0.75,1.374995) & (-1.25,2.375005) \end{bmatrix} =$$
$$\begin{bmatrix} (1,22.499910) & (0,38.000090) \\ (0,13.499950) & (1,19.000050) \end{bmatrix}.$$

Example 5.2 *Consider the following fuzzy matrix of order* 2×2 *which its elements are symmetric triangular fuzzy numbers:*

$$\tilde{A} = \begin{bmatrix} (1,2) & (3,4) \\ (1,6) & (3,7) \end{bmatrix}.$$

The equations system that allows the computation of the fuzzy inverse matrix $\tilde{A}$ is as follows:

$$\begin{bmatrix} (1,2) & (3,4) \\ (1,6) & (3,7) \end{bmatrix} \otimes \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{bmatrix} = \begin{bmatrix} \tilde{1} & \tilde{0} \\ \tilde{0} & \tilde{1} \end{bmatrix}.$$

Therefore, we can write the matrix multiplication in the fuzzy linear equation system form as below:

$$\begin{cases} ((1,2) \otimes \tilde{x}_{11}) \oplus ((3,4) \otimes \tilde{x}_{21}) = (1,\delta), \\ ((1,2) \otimes \tilde{x}_{12}) \oplus ((3,4) \otimes \tilde{x}_{22}) = (0,\alpha), \\ ((1,6) \otimes \tilde{x}_{11}) \oplus ((3,7) \otimes \tilde{x}_{21}) = (0,\alpha), \\ ((1,6) \otimes \tilde{x}_{12}) \oplus ((3,7) \otimes \tilde{x}_{22}) = (1,\delta). \end{cases}$$

The above equation system can be written for the given example as follows:

$$\begin{cases} ((1,2) \otimes (x_{11},w_{11})) \oplus ((3,4) \otimes (x_{21},w_{21})) = (1,\delta), \\ ((1,2) \otimes (x_{12},w_{12})) \oplus ((3,4) \otimes (x_{22},w_{22})) = (0,\alpha), \\ ((1,6) \otimes (x_{11},w_{11})) \oplus ((3,7) \otimes (x_{21},w_{21})) = (0,\alpha), \\ ((1,6) \otimes (x_{12},w_{12})) \oplus ((3,7) \otimes (x_{22},w_{22})) = (1,\delta). \end{cases}$$

The equivalence of the above equation system and the following system is found by implementing the approximate fuzzy multiplication:

$$\begin{cases} x_{11} + 3x_{21} = 1, \\ x_{12} + 3x_{22} = 0, \\ x_{11} + 3x_{21} = 0, \\ x_{12} + 3x_{22} = 1, \\ w_{11} + 2x_{11} + 3w_{21} + 4x_{21} = \delta, \\ w_{12} + 2x_{12} + 3w_{22} + 4x_{22} = \alpha, \\ w_{11} + 6x_{11} + 3w_{21} + 7x_{21} = \alpha, \\ w_{12} + 6x_{12} + 3w_{22} + 7x_{22} = \delta. \end{cases}$$

The Gröbner basis for the ideal generated by the above system in the ring $R = \mathbb{R}[x_{11}, x_{21}, x_{12}, x_{22}, w_{11}, w_{21}, w_{12}, w_{22}]$, w.r.t the graded lexicographic order is as: $\{1\}$. Therefore, with attention to Theorem 4.7, the fuzzy matrix is not invertible.

VI. CONCLUSION

This paper proffers an innovative approach, which was founded on Gröbner bases profits for acquiring a fuzzy inverse matrix including fuzzy numbers of LR type. The problem of computing inverse matrix was converted to solving a system of polynomial equations in fuzzy case in which the right-hand sides and all unknowns, coefficients are fuzzy numbers. The sufficient and necessary condition for invertibility of the fuzzy matrix was discussed. Finally, an algorithm was designed to obtain the fuzzy inverse matrix on the base of Gröbner basis.

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AUTHORS PROFILE

Hamed Farahani received his B.Sc. and M.Sc. degrees in mathematics from Damghan University, in 2004 and Institute for Advanced Studies in Basic Sciences (IASBS) in 2006. He received his doctor's degree in Computer Algebra from Damghan University, in 2014. He is an assistant professor at School of Maritime Science, Chabahar Maritime University (CMU) since 2008. His current research focuses on Groebner basis and its applications.

M. J. Ebadi received his BSc degree in applied mathematics from Bahonar University of Kerman, Iran, in 2003, and his MSc and Ph.D. degrees in applied mathematics from S&B University and Yazd University, Iran, in 2006 and 2018, respectively. In 2006, he joined the department of basic sciences at Chabahar Maritime University, Iran, as a faculty member. He is serving more than 15 international reputed journals as a reviewer. His research interests include neural networks, deep learning, numerical optimization, optimal control, numerical analysis, EEG signals classification, and fuzzy optimization.

Hossein Jafari received his BSc and MSc degrees in Statistics, from Shiraz University, Iran, in 1990 and 1994 respectively, and his Ph.D. degree in mathematics from Mathematics, University of Barcelona, Spain in 2012. In 1994, he joined the department of basic sciences at Chabahar Maritime University, Iran, as a faculty member. His research interests include stochastic analysis, Malliavin calculus, financial mathematics, and fuzzy theory.