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Unique Common Fixed Point Results with Application in Fuzzy Metric Spaces

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***Abstract:** Fixed points in fuzzy theory plays a very significant task in the field of mathematical and computer sciences. This theory has numerous applications in the area of communication and technology, computer science and information security and many others. Many researchers established the results of point for an assortment of mappings in different metric spaces. In present paper, our aim is to establish a common fixed point theorem satisfying generalized contraction in FMS. We also bestow an application for integral type contraction.*

***Keywords:** Integral equations, fixed point, fuzzy metric spaces (FMS), contraction mapping. :54H25, 47S40, 54A40.*

I. INTRODUCTION

L. A. Zadeh [5] was the first person who gave the concept of fuzzy set. This introduction did wonders in the development of many mathematical models and set a base for fuzzy mathematics. Since the early seventy of 20th century, by using fuzzy concept, many results on fixed point for single as well as set valued mappings are proved. In 1975, Kramosil and Michlek [3] gave the concept of FMS. Later in 1988, M. Grabiec [8] proved a fixed point theorem in FMS as follows:

Theorem 1. [8] Let $(E, L, *)$ is a complete FMS such that

1. $\lim_{\alpha\to\infty} L(x, y, \alpha) = 1$,
2. $L(Cx, Cy, m\alpha) \geq L(x, y, \alpha)$, $\forall \; x, y, \in E$, $m \in (0,1)$.

Then a unique fixed point exist for C.

George et al.[1] tailored the concept of FMS using continuous α -norm. By this definition, they have also introduced a Hausdorff topology on such FMS. Now a days, it has been widely used by various researchers in their respective field of researches. Some other fixed point outcomes were extended to FMS by different researchers such as Subrahmanyam [10], Vasuki [11], Fang [4], Gupta and Mani [13, 14].

Jungck [2] introduced the idea of compatible mappings and using this property he proved many common fixed point theorems. After that, Mishra et al. [12] developed compatible mappings in FMS.

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Aamri et al.[6] widespread the concept of (E.A) property and discussed several results.

Definition 1. [12] Two mappings C and D of a FMS $(E, L*)$ into itself are said to be compatible maps if $\lim_{n\to\infty} L(CDx_n, DCx_n, \alpha) = 1$, $\forall \; \alpha > 0$, where $\{x_n\} \in E$ such that $\lim_{n\to\infty} Cx_n = \lim_{n\to\infty} Cx_n = u \in E$.

Definition 2.[6] Self mappings T and R of a FMS are said to satisfies property (E.A.)if there exist a sequence $\{x_n\} \in E$ such that $\lim_{n\to\infty} Rx_n = \lim_{n\to\infty} Tx_n = x_0$ for some $x_0 \in E$.

Definition 3. [9] Let $\xi : [0,1] \to [0,1]$ are such that

1. ξ is non-decreasing,
2. $\xi(\alpha) > \alpha$, $\forall \; \alpha \in (0,1]$ and $\xi(\alpha) = \alpha$ if and only if $\alpha = 1$.

II. RESULTS

Theorem 2. Let C, D, T and R are self maps of a FMS $(E, L, *)$ such that for each $x \neq y \in E$ and $\alpha > 0$

1. (C,T) or (D,R) satisfies E.A property with

$$L(Cx, Dy, \alpha) > \xi\left(\min\left\{\begin{array}{l} L(Tx, Ry, \alpha), L(Tx, Dy, \alpha), \\ L(Ry, Dy, \alpha), \\ \dfrac{L(Cx, Tx, \alpha) + L(Cx, Ry, \alpha)}{2}, \\ \dfrac{L(Dy, Ry, \alpha).L(Dy, Ry, \alpha)}{L(Tx, Ry, \alpha)} \end{array}\right\}\right) \quad (1)$$

2. $C(E) \subset R(E)$ and $D(E) \subset T(E)$.
3. (C,T) and (D,R) are weakly compatible.

If any of the ranges of C, D, T and R is a complete subspace of E, then there exist a unique $w \in E$ such that $Cw = Dw = Tw = Rw = w$.

Proof. Since (D,R) assure the property (E.A), thus there exist $\{x_n\} \in E$ such that $\lim_{n\to\infty} Dx_n = \lim_{n\to\infty} Rx_n = w$ for some $w \in E$.

As $D(E) \subset T(E)$, then for some sequence $\{y_n\}$ in E, $Dx_n = Ty_n = w$. Thus $\lim_{n\to\infty} Ty_n = w$.

Next, we prove that $\lim_{n\to\infty} Cy_n = w$.

Thus from equation (1), we get

$L(Cy_n, w, \alpha) = L(Cy_n, Dx_n, \alpha)$

$$> \xi\left(\min\left\{\begin{array}{l} L(Ty_n,Rx_n,\alpha),L(Ty_n,Dx_n,\alpha),\\ L(Rx_n,Dx_n,\alpha),\\ \dfrac{L(Ty_n,Dx_n,\alpha)+L(Rx_n,Dx_n,\alpha)}{2},\\ \dfrac{L(Ty_n,Rx_n,\alpha).L(Rx_n,Dx_n,\alpha)}{L(Ty_n,Dx_n,\alpha)}\end{array}\right\}\right).$$

Taking $\lim_{n\to\infty}$, we have

$$\lim_{n\to\infty} L(Cy_n,w,\alpha) > \xi\left(\min\left\{\begin{array}{l} L(w,w,\alpha),L(w,w,\alpha),\\ L(w,w,\alpha),\\ \dfrac{L(w,w,\alpha)+L(w,w,\alpha)}{2},\\ \dfrac{L(w,w,\alpha).L(w,w,\alpha)}{L(w,w,\alpha)}\end{array}\right\}\right) > 1.$$

Therefore, $\lim_{n\to\infty} Cy_n = w = \lim_{n\to\infty} Dx_n$.

Suppose that $T(E)$ is a complete subspace of E, then $Tu = w$, for at least one $u \in E$. Thus we have

$$\lim_{n\to\infty} Cy_n = w = \lim_{n\to\infty} Dx_n = \lim_{n\to\infty} Ty_n = Tu. \qquad (2)$$

Next we prove that $Cu = Tu$.

Consider,

$$\lim_{n\to\infty} L(Cu,Dx_n,\alpha) > \xi\left(\min\left\{\begin{array}{l} L(Tu,Tu,\alpha),L(Tu,Tu,\alpha),\\ L(Tu,Tu,\alpha),\\ \dfrac{L(Tu,Tu,\alpha)+L(Tu,Tu,\alpha)}{2},\\ \dfrac{L(Tu,Tu,\alpha).L(Tu,Tu,\alpha)}{L(Tu,Tu,\alpha)}\end{array}\right\}\right)$$

$$> \xi(\min\{1,1,1,1,1\}) = 1.$$

Thus, $lim_{n\to\infty} L(Cu,Tu,\alpha) = 1$ *i.e* $Cu = Tu$. Thus by using weak compatibility of C and T implies that $CTu = TCu$. This implies that $CCu = CTu = TCu = TTu$. But, as $C(E) \subset R(E)$, $\exists\ v \in E$ such that $Cu = Rv$.

Similarly, one can show that $Rv = Dv$. But $Rv = Cu$, therefore $Rv = Dv$. This implies that $Cu = Tu = Rv = Dv$. Since D and R are weakly compatible. therefore, DRv = RDv = RRv = DDv.

Next we claim $CCu = Cu$. For this, let $CCu \neq Cu$.

From eq(1), we have

$L(Cu,CCu,\alpha) = L(CCu,Dv,\alpha)$

$$> \xi\left(min\left\{\begin{array}{l} L(TCu,Rv,\alpha),L(TCu,Dv,\alpha),\\ L(Rv,Dv,\alpha),\\ \dfrac{L(TCu,Dv,\alpha)+L(Rv,Dv,\alpha)}{2},\\ \dfrac{L(TCu,Rv,\alpha).L(Rv,Dv,\alpha)}{L(TCu,Dv,\alpha)}\end{array}\right\}\right)$$

$$> \frac{L(CCu,Cu,\alpha)+1}{2} > 1$$

Thus $CCu = Cu$. Similarly, one can prove that $DCu = Cu$, $RCu = Cu$ and $TCu = Cu$. This proves our claim.

Uniqueness: Let w and v be the fixed point for C, D, T and R. Thus,

$Cw = Dw = Tw = Rw = w$ and $Cv = Dv = Tv = Rv = v$

Now $L(w,v,\alpha) = L(Cw,Dv,\alpha)$

$$> \xi\left(\min\left\{\begin{array}{l} L(Tw,Rv,\alpha),L(Tw,Dv,\alpha),\\ L(Rv,Dv,\alpha),\\ \dfrac{L(Tw,Dv,\alpha)+L(Rv,Dv,\alpha)}{2},\\ \dfrac{L(Tw,Rv,\alpha).L(Rv,Dv,\alpha)}{L(Tw,Dv,\alpha)}\end{array}\right\}\right)$$

$$> \xi\left(\min\left\{\begin{array}{l} L(w,v,\alpha),L(w,v,\alpha),L(v,v,\alpha)\\ \dfrac{L(w,v,\alpha)+L(v,v,\alpha)}{2},\\ \dfrac{L(w,v,\alpha).L(v,v,\alpha)}{L(w,v,\alpha)}\end{array}\right\}\right)$$

$$> \xi\left(\min\left\{L(w,v,\alpha),\frac{L(w,v,\alpha)+1}{2}\right\}\right)$$

Now if $L(w,v,\alpha) < \dfrac{L(w,v,\alpha)+1}{2}$, then result follows directly.

But if $L(w,v,\alpha) > \dfrac{L(w,v,\alpha)+1}{2}$, then

$$L(w,v,\alpha) > \xi\left(\frac{L(w,v,\alpha)+1}{2}\right) > \frac{L(w,v,\alpha)+1}{2} > 1,$$

Thus, $w = v$. This proves the uniqueness and hence the result.

Now, we give an application based to our result. Let $\Psi:[0,\infty)\to[0,\infty)$, $\Psi(\alpha)=\int_0^{\alpha}\varphi(\alpha)\,dt \quad \forall \quad \alpha>0$, be a continuous and nondecreasing function. Also, $\varphi(\epsilon)>0$ for each $\epsilon>0$, and $\varphi(\alpha)=0$ iff $\alpha=0$.

Theorem 3. Let C, D, T and R be self-maps of a FMS $(E,L,*)$ such that for each $x \neq y \in E$ and $\alpha > 0$

1. (C,T) or (D,R) satisfies E.A property,
2. $\int_0^{L(Cx,Dy,kt)}\varphi(\alpha)\,dt \geq \int_0^{\lambda(x,y,\alpha)}\varphi(\alpha)\,dt$ where $\varphi \in \Psi$ and

$$\lambda(x,y,\alpha) = \xi\left(\min\left\{\begin{array}{l} \{L(Tx,Ry,\alpha),L(Tx,Dy,\alpha),\\ L(Ry,Dy,\alpha),\\ \dfrac{L(Tx,Dy,\alpha)+L(Ry,Dy,\alpha)}{2},\\ \dfrac{L(Tx,Ry,\alpha).L(Ry,Dy,\alpha)}{L(Tx,Dy,\alpha)}\end{array}\right\}\right)$$

3. $C(E) \subset R(E)$ and $D(E) \subset T(E)$
4. (C,T) and (D,R) are weakly compatible

If at least one of the images of C, D, T and R is complete subspace of E, then there exist a unique $w \in E$ such that $Cw = Dw = Tw = Rw = w$.

Proof. Choose $\varphi(\alpha)=1$ and using Theorem 2 , one can get the outcome.

III. CONCLUSION AND FUTURE SCOPE

Since its beginning in 1965, fuzzy sets theory has been highly developed in various disciplines. Applications on fuzziness can be found, for example, in computer science, artificial intelligence and medicine. Fuzzy set theory has been studied in various directions by different authors. FMS is one of the influential and remarkable areas of fuzzy theory. In this study, some fixed point results by using iterative and convergence criteria have been established in FMS. For the future research, further execution and juxtaposition of the results proved in this article with similar existing result is an open area.

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