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Stability of Certain Discrete Fractional Equations of Order $1 < \vartheta \le 2$

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Abstract: The equations used to model a real life event is often nonlinear due to the fact that the linear terms fails to bring out various characteristics. Obtaining the exact solution of the nonlinear equation is complicated which makes one to deal with the qualitative properties of the equation. Simple Harmonic Motion (SHM) which is periodic in nature has numerous applications in clock, car shock absorbers, earth quake, heart beat etc and plays a important role in modeling the motion of a particle. In this paper, we consider a initial value discrete fractional equation. The Hyers-Ulam stability and Hyers-Ulam-Mittag-Leffler stability is established for the equation. The stability of discrete fractional simple pendulum equation is established with simulations.

Keywords : Fractional order, Discrete, Mittag-Leffler function, Hyers Ulam Stability.

I. INTRODUCTION

Fractional Calculus, a mathematical branch investigating non integer order derivatives dates back to the same time when the integer order calculus was established. The development in theory of fractional differential equations has justified the increasing interest among the researchers towards the field. The application of fractional differential equations in engineering has gained significant attention from the world of researchers and scientists. In understanding the applications, the qualitative theories of the equation like stability, oscillation, asymptotic properties etc play a predominant role. Analysis of the arbitrary order differential equation with non local and weakly singular kernals prove to be challenging than that of integer order differential equations. Therefore, the evolution of equivalent qualitative theory for the fractional equations are quite slow. Despite the progress in the field of continuous fractional calculus, its counterpart discrete fractional calculus had no parallel development. It is just over a decade where the theory for the discrete case was initiated.

The works in discrete fractional calculus by Atici and Eloe [[1], [2], [3], [4]], Goodrich [12] and Miller and Ross [17] are the basis for the recent developments in the field.

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The stability of the discrete fractional equations are discussed by Fulai chen in [[8], [9]]. The Hyer's answer to Ulam's question which he raised during a talk in 1940 was origin to the study of stability of functional equations[[13], [18]]. The Ulam stability concept has been extended in different direction. Hyers-Ulam stability of the fractional differential equations has been studied in [[5],[14], [16]]. There are only few works carried out in case of Ulam stability of fractional difference equations [[6], [15]]. The Ulam stability of the boundary value discrete fractional equation was studied in [10]. Motivated by the work of Churong Chen in [7], we here consider the discrete fractional undamped duffing equation with forcing term.

$$\begin{cases} \Delta_*^{\vartheta}[\phi(n)] = f(n+\vartheta,\phi(n+\vartheta)), n \in Q, 1<\vartheta\le 2 \\ \qquad \phi(0) = A, \Delta(\phi(0)) = B \end{cases} \tag{1}$$

where Δ_*^{ϑ} is a Caputo like difference operator, $f: Q \to \mathbb{R}$. Denote $Q:[i, i+T]\cap \mathbb{N}_i$, where $T \in \mathbb{N}$ and $\mathbb{N}_i = \{i, i+1, \ldots\}, i \in \mathbb{R}$.

The paper is structured as follows. Mathematical background with some basic definitions and lemmas are imparted in section 2. Hyers-Ulam stability and Hyers-Ulam-Mittag-Leffler stability is presented in section 3 and 4 respectively. Numerical example with simulation is provided in Section 5.

II. MATHEMATICAL BACKGROUND

Definition 1 *[3] ($\alpha-th$ Fractional Sum) Let $\alpha>0$. The $\alpha-th$ fractional sum of g is defined by*

$$\Delta^{-\alpha} g(n) = \frac{1}{\Gamma(\alpha)} \sum_{s=i}^{n-\alpha} (n-s-1)^{(\alpha-1)} g(s), \tag{2}$$

where g is defined for $s = i mod(1)$ and $\Delta^{-\alpha} g$ is defined for $n = (i+\alpha) mod(1)$, and $n^{(\alpha)} = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}$.

Definition 2 *[3] Let $\mu>0$ and $k-1<\mu<k$, where k denotes a positive integer, $k=\lceil \mu \rceil$, $\lceil . \rceil$ denotes the ceiling of number. Set $\alpha = k-\mu$. The $\mu-th$ fractional Caputo like difference is defined as*

$$\Delta_*^{\mu} g(n) = \Delta^{-\alpha}\left(\Delta^k g(n)\right) = \frac{1}{\Gamma(\alpha)} \sum_{s=i}^{n-\alpha} (n-s-1)^{(\alpha-1)} \left(\Delta^k g\right)(s) \quad \forall n \in N_{i+\alpha}$$

where Δ^k is the $k-th$ order forward difference operator.

Lemma 3 *[11] For $\mu>0$, μ is non- integer, $k=\lceil \mu \rceil, \alpha = k-\mu$, it holds*

$$f(n)=\sum_{m=0}^{k-1}\frac{(n-i)^{(m)}}{m!}\Delta^m[f(a)]+\frac{1}{\Gamma(\mu)}\sum_{s=i+\alpha}^{n-\mu}(n-s-1)^{(\mu-1)}\Delta_*^{\mu}[f(s)],$$

where f is defined on N_a with $a\in\mathbb{Z}^+$.

In particular, when $1<\mu<2$ and $a=0$, we have

$$f(n)=[f(0)+n\Delta(f(0))]+\frac{1}{\Gamma(\mu)}\sum_{s=1-\mu}^{n-\mu}(n-s-1)^{(\mu-1)}\Delta_*^{\mu}[f(s)] \quad (3)$$

where f is defined on N_1 and Δ_*^{μ} is defined on $N_{1-\mu}$.

Lemma 4 *A solution $\phi(n):Q\to\mathbb{R}$ of (1) iff $\phi(n)$ is a solution of the fractional Taylor's difference formula given by*

$$\phi(n)=A+nB+\frac{1}{\Gamma(\vartheta)}\sum_{s=2-\vartheta}^{n-\vartheta}(n-s-1)^{(\vartheta-1)}\left(f(s+\vartheta,\phi(s+\vartheta))\right),n\in Q \quad (4)$$

Proof. Suppose that $\phi(n)$ is a solution of (1), we have from (3)

$$\phi(n)=[\phi(0)+n\Delta(\phi(0))]+\frac{1}{\Gamma(\vartheta)}\sum_{s=2-\vartheta}^{n-\vartheta}(n-s-1)^{(\vartheta-1)}\Delta_*^{\vartheta}[\phi(s)]$$

$$\phi(n)=[A+nB]+\frac{1}{\Gamma(\vartheta)}\sum_{s=2-\vartheta}^{n-\vartheta}(n-s-1)^{(\vartheta-1)}\left(f(s+\vartheta,\phi(s+\vartheta))\right)$$

clearly, (4) holds. Conversely, if $\phi(n)$ is solution of (4), comparing (3) and (4) yields,

$$\sum_{s=2-\vartheta}^{n-\vartheta}(n-s-1)^{(\vartheta-1)}\Delta_*^{\vartheta}[\phi(s)]=\sum_{s=2-\vartheta}^{n-\vartheta}(n-s-1)^{(\vartheta-1)}\left(f(s+\vartheta,\phi(s+\vartheta))\right),$$

which takes the form

$$\sum_{s=2-\vartheta}^{n-\vartheta}(n-s-1)^{(\vartheta-1)}[\Delta_*^{\vartheta}[\phi(s)]-\left(f(s+\vartheta,\phi(s+\vartheta))\right)]=0,n\in Q \quad (5)$$

For any $n\in Q$, we have

$$\Delta_*^{\vartheta}[\phi(n)]=f(n+\vartheta,\phi(n+\vartheta)) \quad n\in Q \quad (6)$$

Thus, it is clear that $\phi(n)$ is solution of (1). This completes the proof.

Lemma 5 *One has*

$$\sum_{s=2-\vartheta}^{n-\vartheta}(n-s-1)^{(\vartheta-1)}=\frac{(n+\vartheta-2)^{(\vartheta)}}{\vartheta} \quad (7)$$

Proof. For $c>p,c,p\in\mathbb{R},c>-1,p>-1,$ we have

$$\frac{\Gamma(c+1)}{\Gamma(c-p+1)}=\frac{1}{p+1}\left[\frac{\Gamma(c+2)}{\Gamma(c-p+1)}-\frac{\Gamma(c+1)}{\Gamma(c-p)}\right], \quad (8)$$

$$\sum_{s=2-\vartheta}^{n-\vartheta}(n-s-1)^{(\vartheta-1)}=\sum_{s=2-\vartheta}^{n-\vartheta}\frac{\Gamma(n-s)}{\Gamma(n-s-\alpha+1)}$$
$$=\sum_{s=2-\vartheta}^{n-\vartheta-1}\frac{\Gamma(n-s)}{\Gamma(n-s-\alpha+1)}+\Gamma(\vartheta)$$
$$=\sum_{s=2-\vartheta}^{n-\vartheta-1}\frac{1}{\vartheta}\left[\frac{\Gamma(n-s+1)}{\Gamma(n-s-\alpha+1)}-\frac{\Gamma(n-s)}{\Gamma(n-s-\alpha)}\right]+\Gamma(\vartheta)$$
$$=\frac{1}{\vartheta}\left[\frac{\Gamma(n+\alpha-1)}{\Gamma(n-1)}-\Gamma(\vartheta+1)\right]$$
$$=\frac{(n+\vartheta-2)^{(\vartheta)}}{\vartheta}$$

Lemma 6 *[3] Let $\vartheta\neq 1$ and assume $\mu+\vartheta+1$ is a positive integer, then*

$$\Delta^{-\vartheta}n^{\mu}=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\vartheta+1)}n^{(\mu+\vartheta)}, \qquad t\in Q. \quad (9)$$

III. HYERS-ULAM STABILITY

Definition 7 *The discrete fractional equation (1) with initial condition is Hyers-Ulam Stable, if there exists a positive constant W with the property: For every $\varepsilon>0$, $\psi\in\mathbb{R}$ satisfies the inequality*

$$\left|\Delta_*^{\vartheta}[\psi(n)]+f(n+\vartheta,\psi(n+\vartheta))\right|\le\varepsilon \quad n\in Q \quad (10)$$

with initial condition $\psi(0)=A,\Delta(\psi(0))=B$ then there exists solution $\phi(n)$ of (1) such that $|\psi(n)-\phi(n)|\le W\varepsilon$. Here W is the Hyers-Ulam stability constant.

Remark 1:

A function $\psi(n)\in\mathbb{R}$ solves the inequality (10) iff there exists $g:Q\times\mathbb{R}\to\mathbb{R}$ such that

1. $|g(n+\vartheta,\psi(n+\vartheta))|\le\varepsilon \qquad n\in Q.$
2. $\Delta_*^{\vartheta}[\psi(n)]=f(n+\vartheta,\psi(n+\vartheta))+g(n+\vartheta,\psi(n+\vartheta)).$

Lemma 8 *If $\psi(n)$ solves (10) then*

$$\left|\psi(n)-\psi(0)-n\Delta\psi(0)-\frac{1}{\Gamma(\vartheta)}\sum_{s=2-\vartheta}^{n-\vartheta}(n-s-1)^{(\vartheta-1)}(f(s+\vartheta,\psi(s+\vartheta)))\right|$$
$$\le\varepsilon\frac{(T+\vartheta-2)^{(\vartheta)}}{\Gamma(\vartheta+1)} \quad n\in Q \quad (11)$$

Proof. If ψ solves (10), using Remark 1 and (3) we can obtain that the solution to $R2$ that satisfies

$$\psi(n)=\psi(0)+n\Delta\psi(0)+\frac{1}{\Gamma(\vartheta)}\sum_{s=2-\vartheta}^{n-\vartheta}(n-s-1)^{(\vartheta-1)}\left(g(s+\vartheta,\psi(s+\vartheta))+f(s+\vartheta,\psi(s+\vartheta))\right),n\in Q$$

Hence,

$$\left|\psi(n)-\psi(0)-n\Delta\psi(0)-\Delta^{-\vartheta}\left(f(n+\vartheta,\psi(n+\vartheta))\right)\right|$$
$$=\left|\Delta^{-\vartheta}g(n+\vartheta,\psi(n+\vartheta))\right|$$
$$\le\frac{1}{\Gamma(\vartheta)}\sum_{s=2-\vartheta}^{n-\vartheta}(n-s-1)^{(\vartheta-1)}\left|g(s+\vartheta,\psi(s+\vartheta))\right|$$
$$\le\varepsilon\frac{1}{\Gamma(\vartheta)}\sum_{s=2-\vartheta}^{t-\vartheta}(n-s-1)^{(\vartheta-1)}$$
$$\le\varepsilon\frac{(T+\vartheta-2)^{(\vartheta)}}{\Gamma(\vartheta+1)}$$

This completes the proof.

Prior to proving the Hyers-Ulam Stability of (1), we make the following assumption

1. Let the function $F(n,\phi)$ is Lipschitz continuous, ie, for all $\phi,\psi\in\mathbb{R}$ and $n\in Q$ there exists $L>0$ such that

$$|F(n,\phi)-F(n,\psi)|\le L|\phi-\psi| \quad (12)$$

Theorem 9 *Assume that $(H1)$ hold. Let $\psi\in\mathbb{R}$ solve (10) for some $\varepsilon>0$ and let $\phi\in\mathbb{R}$ be the solution of*

$$\begin{cases} \Delta_*^{\vartheta}[\phi(n)] = f(n+\vartheta, \phi(n+\vartheta)), n \in Q, 1 < \vartheta \le 2 \\ \phi(0) = \psi(0), \Delta(\phi(0)) = \Delta(\psi(0)) \end{cases} \quad (13)$$

then (1) is Hyers-Ulam Stable provided

$$\Gamma(T+\vartheta-1)L < \Gamma(\vartheta+1)\Gamma(T-1)$$

Proof. From Lemma (4), the solution $\phi(n)$ of (13) satisfies

$$\phi(n) = \psi(0) + n\Delta\psi(0) + \frac{1}{\Gamma(\vartheta)} \sum_{s=2-\vartheta}^{n-\vartheta} (n-s-1)^{(\vartheta-1)} \left(f(s+\vartheta, \phi(s+\vartheta)) \right), n \in Q$$

Therefore

$$\begin{aligned} |\psi(n)-\phi(n)| &= \left|\psi(n)-\psi(0)-n\Delta\psi(0)-\Delta^{-\vartheta}\left(f(n+\vartheta,\phi(n+\vartheta))\right)\right| \\ &= |\psi(n)-\psi(0)-n\Delta\psi(0)-\Delta^{-\vartheta}\left(f(n+\vartheta,\psi(n+\vartheta))\right) \\ &\quad -\Delta^{-\vartheta}\left(f(n+\vartheta,\phi(n+\vartheta))\right)+\Delta^{-\vartheta}\left(f(n+\vartheta,\psi(n+\vartheta))\right)| \\ &\le \left|\psi(n)-\psi(0)-n\Delta\psi(0)-\Delta^{-\vartheta}\left(f(n+\vartheta,\psi(n+\vartheta))\right)\right| \\ &\quad +L\Delta^{-\vartheta}\left(|\psi-\phi|\right) \\ &\le \varepsilon\frac{(T+\vartheta-2)^{(\vartheta)}}{\Gamma(\vartheta+1)} + L\Delta^{-\vartheta}|\psi-\phi| \\ &\le W\varepsilon \end{aligned}$$

where $\xi = \dfrac{L(T+\vartheta-2)^{(\vartheta)}}{\Gamma(\vartheta+1)}$

Thus, it is clear that (1) is Hyers-Ulam stable with Hyers-Ulam stability constant $W = \dfrac{(T+\vartheta-2)^{(\vartheta)}}{\Gamma(\vartheta+1)(1-\xi)}$. This completes the proof.

IV. HYERS-ULAM-MITTAG-LEFFLER STABILITY

Definition 10 *The initial value discrete fractional equation (1) is Hyers-Ulam-Mittag-Leffler Stable with Mittag Leffler function $F_{\vartheta}(\lambda,n)$, if there exists a positive constant W_1 with the property: For every $\varepsilon > 0$, $\psi(n) \in \mathbb{R}$ satisfies the inequality*

$$\left|\Delta_*^{\vartheta}[\psi(n)] - f(n+\vartheta,\psi(n+\vartheta))\right| \le F_{\vartheta}(\lambda,n)\varepsilon \quad n \in Q \quad (14)$$

with initial condition $\psi(0) = A, \Delta(\psi(0)) = B$ then there exists solution $\phi(n)$ of (1) such that $|\psi(n)-\phi(n)| \le W_1\varepsilon F_{\vartheta}(\lambda,n)$. Here W_1 is the stability constant.

Remark 2:

A function $\psi(n) \in \mathbb{R}$ solves the inequality (10) iff there exists $g: Q \times \mathbb{R} \to \mathbb{R}$ such that

1. $|g(n+\vartheta,\psi(n+\vartheta))| \le \varepsilon F_{\vartheta}(\lambda,n) \qquad n \in Q.$
2. $\Delta_*^{\vartheta}[\psi(n)] = f(n+\vartheta,\psi(n+\vartheta)) + g(n+\vartheta,\psi(n+\vartheta)).$

Lemma 11 *If $\psi(n)$ solves (16) then*

$$\left|\psi(n)-\psi(0)-n\Delta\psi(0)-\frac{1}{\Gamma(\vartheta)}\sum_{s=2-\vartheta}^{n-\vartheta}(n-s-1)^{(\vartheta-1)}(f(s+\vartheta,\phi(s+\vartheta)))\right| \le \varepsilon F_{\vartheta}(\lambda,n) \quad n \in Q \quad (15)$$

Proof. If ψ solves (16), using Remark 1 and (3) we can obtain that the solution to $R2$ that satisfies

$$\psi(n) = \psi(0) + n\Delta\psi(0) + \frac{1}{\Gamma(\vartheta)}\sum_{s=2-\vartheta}^{n-\vartheta}(n-s-1)^{(\vartheta-1)}\left(g(s+\vartheta,\psi(s+\vartheta)) + f(s+\vartheta,\psi(s+\vartheta))\right), n \in Q$$

Hence,

$$\begin{aligned} &\left|\psi(n)-\psi(0)-n\Delta\psi(0)-\Delta^{-\vartheta}\left(f(n+\vartheta,\psi(n+\vartheta))\right)\right| \\ &= \left|\Delta^{-\vartheta}g(n+\vartheta,\psi(n+\vartheta))\right| \\ &\le \Delta^{-\vartheta}\left|g(n+\vartheta,\psi(n+\vartheta))\right| \\ &\le \varepsilon\Delta^{-\vartheta}F_{\vartheta}(\lambda,n) \\ &\le \frac{\varepsilon}{\lambda}F_{\vartheta}(\lambda,n) \end{aligned}$$

This completes the proof.

Theorem 12 *Assume that* $(H1)$ *hold. Let* $\psi \in \mathbb{R}$ *solve (16) for some* $\varepsilon > 0$ *and let* $\phi \in \mathbb{R}$ *be the solution of (13) then (1) is Hyers-Ulam-Mittag-Leffler Stable provided*

$$(T+\vartheta-2)^{(\vartheta)}[L] < \Gamma(\vartheta+1)$$

Proof. From Lemma (4), the solution $\phi(n)$ of (13) satisfies

$$\phi(n) = \psi(0) + n\Delta\psi(0) + \frac{1}{\Gamma(\vartheta)}\sum_{s=2-\vartheta}^{n-\vartheta}(n-s-1)^{(\vartheta-1)}\left(f(s+\vartheta,\phi(s+\vartheta))\right), n \in Q$$

Therefore,

$$\begin{aligned} |\psi(n)-\phi(n)| &= \left|\psi(n)-\psi(0)-n\Delta\psi(0)-\Delta^{-\vartheta}\left(f(n+\vartheta,\phi(n+\vartheta))\right)\right| \\ &= |\psi(n)-\psi(0)-n\Delta\psi(0)-\Delta^{-\vartheta}f(n+\vartheta,\psi(n+\vartheta)) \\ &\quad -\Delta^{-\vartheta}\left(f(n+\vartheta,\phi(n+\vartheta))\right)+\Delta^{-\vartheta}\left(f(n+\vartheta,\psi(n+\vartheta))\right)| \\ &\le \left|\psi(n)-\psi(0)-n\Delta\psi(0)-\Delta^{-\vartheta}\left(f(n+\vartheta,\psi(n+\vartheta))\right)\right| \\ &\quad +L\Delta^{-\vartheta}\left(|\psi-\phi|\right) \\ &\le \frac{\varepsilon}{\lambda}F_{\vartheta}(\lambda,t) + L\Delta^{-\vartheta}|\psi-\phi| \\ &\le W_1\varepsilon \end{aligned}$$

Thus, it is clear that (1) is Hyers-Ulam-Mittag-Leffler stable with stability constant $W_1 = \dfrac{\varepsilon}{\lambda(1-\xi_1)}F_{\vartheta}(\lambda,n)$ where $\xi_1 = \dfrac{L(T+\vartheta-2)^{(\vartheta)}}{\Gamma(\vartheta+1)}$.

This completes the proof.

V. NUMERICAL EXAMPLE

In classical mechanics, a particle's motion is described by change of its position and velocity with variation in time. Newton's law of motion form the basis for the dynamical representation of these variables in terms of the differential equations as the function of time. Differential equations are used in formulation of fundamental laws in sciences (Physics and Chemistry), modeling of complex biological systems. An object is said to undergo oscillatory motion if it repeats the same movement time and again. These motions continues forever, but the system with oscillatory motion will at some point settles at the equilibrium position. Physical world is full of particles exhibiting oscillatory motions like the suspension bridges undergoing oscillatory motion sometimes resulting in disaster, waves in oceans, solid earth's seismic waves, oscillation of the uranium nucleus before it fissions with carbon-di-oxide.

SHM is a oscillatory motion in which the particle moves about the equilibrium position in same path. SHM is very important in fields of seismo-scope, bungee cords to rubber bands. In nature most of the mechanical vibrations are SHM. \

Example 1 *Consider discrete fractional simple pendulum equation is given by*

$$\begin{cases} \Delta_*^{1.9}[\phi(n)] + (0.09)^2 \sin(\phi(n+1.9)) = 0, n \in [0,15] \cap N_0, 1 < \vartheta \leq 2 \\ \phi(0) = 0, \Delta(\phi(0)) = 0 \end{cases} \tag{16}$$

We shall now prove that (16) is Hyers-Ulam stable. First we shall check that $f(n+1.9,\phi) = (0.09)^2 \sin(\phi)$ satisfies $H1$

$$\left| f(n+1.9,\phi) - f(n+1.9,\psi) \right| = \left| (0.09)^2 \sin(\phi) - (0.09)^2 \sin(\psi) \right|$$
$$\leq (0.09)^2 \left| (\phi - \psi) \right|, \qquad n \in [0,15] \cap N_0$$

Clearly, $f(n+1.9,\phi)$ is Lipschitz continuous for $n \in [0,15] \cap N_0$. From Theorem (9) it is clear that $\xi = 0.7079 < 1$.

Let $\varepsilon = 0.45$ and $\psi(t) = \left(\frac{n}{5}\right)^2, n \in [0,15] \cap N_0$. We now show that the inequality (10) holds.

$$\left| \Delta^{1.9}\psi(n) + f(n+1.9, \psi(n+1.9)) \right|$$
$$= \left| \Delta^{1.9}\psi(n) + (0.09)^2 \sin(\psi(n+1.9)) \right|$$
$$= \left| \Delta^{-0.1}\Delta^2 \left(\left(\frac{n}{5} \right)^2 \right) + (0.09)^2 \sin(\psi(n+1.9)) \right|$$
$$= \left| \Delta^{-0.1} \frac{2}{5} + (0.09)^2 \sin(\psi(n+1.9)) \right|$$
$$\leq \left| 0.43 + (0.09)^2 \sin(\psi(n+1.9)) \right|$$
$$\leq 0.44 < \varepsilon, n \in [0,15] \cap N_0.$$

From Theorem (9), it is clear that (16) is Hyers-Ulam stable with Hyers- ulam stability constant W .

Table 1: Illustration of ρ and ϑ

ϑ	$\omega = 0.05$	$\omega = 0.06$	$\omega = 0.07$	$\omega = 0.08$	$\omega = 0.09$
	ξ	ξ	ξ	ξ	ξ
0.0	0.1134	0.0896	0.0686	0.0504	0.0350
1.1	0.1416	0.1119	0.0857	0.0629	0.0437
1.2	0.1760	0.1390	0.1064	0.0782	0.0543
1.3	0.2175	0.1718	0.1316	0.0967	0.0671
1.4	0.2675	0.2114	0.1618	0.1189	0.0826
1.5	0.3276	0.2589	0.1982	0.1456	0.1011
1.6	0.3996	0.3157	0.2417	0.1776	0.1233
1.7	0.4853	0.3834	0.2936	0.2157	0.1498
1.8	0.5872	0.4640	0.3552	0.2610	0.1812
1.9	0.7079	0.5593	0.4282	0.3146	0.2185
2.0	0.8505	0.6720	0.5145	0.3780	0.2625

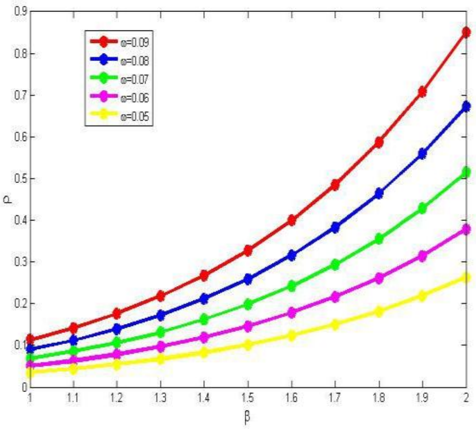

Figure 1: ϑ versus ρ

Table 2: Illustration of ρ and ω

ω	$\vartheta = 1.5$	$\vartheta = 1.6$	$\vartheta = 1.7$	$\vartheta = 1.8$	$\vartheta = 1.9$
	ξ	ξ	ξ	ξ	ξ
0.01	0.0040	0.0049	0.0060	0.0072	0.0087
0.02	0.0162	0.0197	0.0240	0.0290	0.0350
0.03	0.0364	0.0444	0.0539	0.0652	0.0787
0.04	0.0647	0.0789	0.0959	0.1160	0.1398
0.05	0.1011	0.1233	0.1498	0.1812	0.2185
0.06	0.1456	0.1776	0.2157	0.2610	0.3146
0.07	0.1982	0.2417	0.2936	0.3552	0.4282
0.08	0.2589	0.3157	0.3834	0.4640	0.5593
0.09	0.3276	0.3996	0.4853	0.5872	0.7079

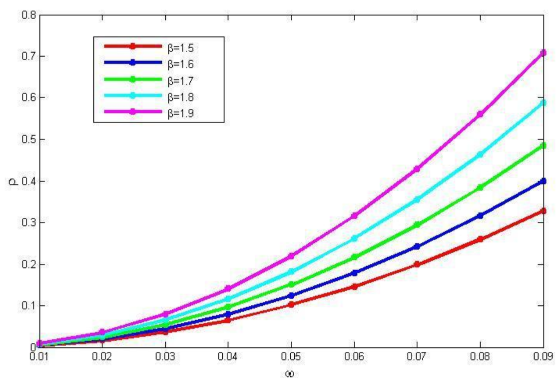

Figure 2: ϑ versus ρ

The values of ξ for different fractional order of the equation fixing the value ω from $0.05 - 0.09$ are tabulated in Table (1) and corresponding values of ϑ and ξ are plotted in Figure 1. Table (1) illustrates the change of ξ for different ω and fixed fractional order between $\vartheta = [1,2]$ with Figure 2 plotting the relation between ω and ϑ. Figure 1 and Figure 2 ensures that the discrete fractional pendulum equation is Hyers-Ulam stable for $1 < \vartheta < 2$ and values of $\omega \leq 0.09$.

VI. CONCLUSION

Hyers-Ulam and Hyers-Ulam-Mittag-Leffler Stability of the initial value discrete fractional equation are established and a numerical example investigated the discrete fractional simple pendulum. Numerical simulations are carried out for the stability results illustrating the effects of the fractional order on the stability conditions. The values are tabulated and plotted.

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