International Journal of Innovative Technology and Exploring Engineering (IJITEE)
ISSN: 2278-3075 (Online), Volume-9 Issue-2, December 2019

An $M/M/1$ Queueing System with Server on Attacks and Repair

Jagadeesan Viswanath, Swaminathan Udayabaskaran

Abstract*: We consider a $M/M/1$ queueing model of a communication system subject to random attacks on service station. In such attack system, the time interval between any two attacks is an exponentially distributed random variable with mean $1/\gamma$. When the server fails by an attack, any customer getting service at that time becomes damaged. The failed server is immediately taken for repair and the damaged customer is washed out. Customers in the queue waiting for service are not washed out. The repair time of the server undergoing repair is assumed to be an exponentially distributed random variable with mean $1/\eta$. During repair time, the customers are allowed to wait for service maintain First Come First Served order. After the completion of repair, the server returns to the work-station immediately without any delay, even if there is no customer to render service. We derive explicit form of the state probabilities of the attack system in the long run. We also obtain measures of system performance and the model is validated by a numerical illustration.

***Keywords* : About four key words or phrases in alphabetical order, separated by commas.**

I. INTRODUCTION

Queueing systems subject to disasters and repair have been studied by several authors in the past (see, for example, [3], [1], [2], [4] and [6]). In [3], a single server queue has been considered where the server fails and all customers are washed out due to the occurrence of catastrophes, the failed server is taken for repair and no customer is entertained to the system during the repair time of the server. In [1] and [2], the same situation as in [1] has been adopted along with a new concept that new customers are allowed to join during repair time with a freedom to accept a loyalty reward for joining. In [4], a $M_X/G/1$ queueis analyzed where Poisson distributed batch arrival of customers takes place, a multiple vacation policy is adopted, disasters takes place according to a Poisson process affecting the server engaged with a customer only and washing out all customers, the affected server is taken for repair and customers are permitted to join the system during the repair period.

Revised Manuscript Received on December 30, 2019.
* Correspondence Author
Jagadeesan Viswanath*, School of Basic Sciences and Humanities, Vel Tech Rangarajan Dr. SagunthalaR & D Institute of Science and Technology, Chennai, India.
Swaminathan Udayabaskaran, School of Basic Sciences and Humanities, Vel Tech Rangarajan Dr. SagunthalaR & D Institute of Science and Technology, Chennai, India.

In paper [6], a transient analysis of a many-server infinite Markovian queueing system has been analyzed with the occurrence of catastrophes, server failure and repair. The authors of [6] have assumed that, there are c servers in the system; customers arrive according to a Poisson process with rate λ; an arriving customer joins the system with probability 1, if there are not more than $c-1$ customers; otherwise, joins the system with probability β, where $0 < \beta < 1$; catastrophes occur according to a Poisson process; whenever a catastrophe occurs, all customers either undergoing service or waiting for service are washed out, and those servers who were rendering service just before the occurrence of the catastrophe become inoperative and immediately taken for exponential repair; customers are restricted to enter the system, while the server is in repair station. In all the above models, all customers waiting for getting service are washed out whenever a catastrophe occurs. In the present paper, we remove this restrictions and propose a queueing model in which (i) Waiting customers are not washed out whenever an attack (catastrophe) occurs; the catastrophe affects only the server and that customer getting the service; (ii) Customers are allowed to join the system during the repair time of the server.Organization of the paper is as follows:In Section 2, the model is described and a list of notations used in the model is provided and also the governing equations of the system are derived. Section 3 obtains explicit expressions for the stationary probabilities of the states of the system. In section 4, we present the measures of performance of the attack system and highlight them by a numerical illustration. Section 5 provides a conclusion where the novelty of the present paper and its findings are presented.

II. DESCRIPTION AND GOVERNING EQUATIONS OF ATTACK SYSTEM

A $M/M/1$ queueing system is considered where arrival pattern follows Poisson process with arrival rate λ. First-Come-First-Served policy is adopted in serving the customers. Service time follows exponential distribution with mean $1/\mu$. Infinite buffer facility for the customers to wait in order to get service is assumed. Server is subject to failures by random attacks and the time interval between any two attacks is exponentially distribution with mean $1/\gamma$.

When the server fails, the customer (if any getting service) becomes damaged. The failed server is immediately taken for repair and the damaged customer is discarded once for all.

Customers in the buffer remain and are not washed out. The server repair time is an independent random variable which is exponentially distributed with mean $1/\eta$. Customers are permitted to join the system while the server is under repair. Without any delay, the server is ready to work immediately after the repair completion, even if there is no waiting customer in the system.

We follow the notation given below:

a. $X(\tau)$ represents the number of customers (which includes the one who is getting service) in the attack system at time τ.

b. $Y(\tau)$ takes the value 0, if the server is online (not under repair) at time τ. And takes the value 1, if the server is undergoing repair at time τ.

c. $Z(\tau)=\left(X(\tau),Y(\tau)\right)$

d. $l_1(u)\Theta l_2(\tau)=\int_0^{\tau} l_1(u)l_2(\tau-u)du.$

e. $l^*(\tau)=\int_0^{\infty} e^{-s\tau}l(\tau)d\tau.$

As per the assumptions, the stochastic process $\{Z(\tau)\,|\,\tau\geq 0\}$ is identified as a discrete-valuedMarkov process with state space

$$\Omega=\{(j,k)\,|\,j=0,1,2,\ldots;k=0,1\}.$$

To study the process, we define the joint probability function:

$$q(j,k;\tau)=\Pr\{X(\tau)=j,Y(\tau)=k\,|\,X(0)=0,Y(0)=0\},$$
$$(j,k)\in\Omega$$

The state-transition rate diagram is described below:

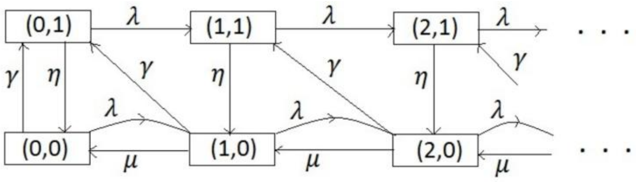

Fig. 1. State-transition rate diagram

Let $f(j,k;\tau)$ denote the first-order product density ([5]) of the stochastic point process of events of entering into the state (j,k). Using probabilistic arguments, following integral equations are derived:

Case (i):

$$q(0,0;\tau)=e^{-(\lambda+\gamma)\tau}+f(0,0;\tau)@e^{-(\lambda+\gamma)\tau}, \quad (1)$$

where $f(0,0;\tau)=q(1,0;\tau)\mu+q(0,1;\tau)\eta.$

Case (ii):

$$q(j,0;\tau)=f(j,0;\tau)@e^{-(\lambda+\gamma+\mu)\tau}, \quad (2)$$

where

$$f(j,0;\tau)=q(j-1,0;\tau)\lambda+q(j+1,0;\tau)\mu+q(j,1;\tau)\eta,$$
$$j=1,2,\ldots.$$

Case (iii):

$$q(0,1;\tau)=f(0,1;\tau)@e^{-(\lambda+\eta)\tau}, \quad (3)$$

where $f(0,1;\tau)=\left\{q(0,0;\tau)+q(1,0;\tau)\right\}\gamma.$

Case (iv):

$$q(j,1;\tau)=f(j,1;\tau)@e^{-(\lambda+\eta)\tau}, \quad (4)$$

where

$$f(j,1;\tau)=q(j-1,1;\tau)\lambda+q(j+1,0;\tau)\gamma,$$
$$j=1,2,\ldots.$$

III. ATTACK SYSTEM'S STEADY STATE ANALYSIS

Consider the steady-state probabilities as follows:

$$\pi(j,k)=\lim_{\tau\to\infty}q(j,k;\tau),(j,k)\in\Omega.$$

Applying final value theorem of Laplacetransform, we get

$$\pi(j,k)=\lim_{\theta\to 0}\theta q^*(j,k;\theta),(j,k)\in\Omega.$$

Applying Laplace Transform on both sides of the equations (1)-(4), we get

$$(\theta+\lambda+\gamma)q^*(0,0;\theta)=1+\mu q^*(1,0;\theta)+\eta q^*(0,1;\theta); \quad (5)$$

$$(\theta+\lambda+\mu+\gamma)q^*(j,0;\theta)=\lambda q^*(j-1,0;\theta)+\mu q^*(j+1,0;\theta)+\eta q^*(j,1;\theta)], \quad (6)$$
$$j=1,2,\ldots;$$

$$(\theta+\lambda+\eta)q^*(0,1;\theta)=\gamma\left\{q^*(0,0;\theta)+q^*(1,0;\theta)\right\}; \quad (7)$$

$$(\theta+\lambda+\eta)q^*(j,1;\theta)=\lambda q^*(j-1,1;\theta)+\gamma q^*(j+1,0;\theta),\ j=1,2,\ldots. \quad (8)$$

Multiplying by θ on both sides of Eqns. (5)-(8) and taking limit as $\theta\to 0$, we get

$$(\lambda+\gamma)\pi(0,0)=\mu\pi(1,0)+\eta\pi(0,1); \quad (9)$$

$$(\lambda+\mu+\gamma)\pi(j,0)=\lambda\pi(j-1,0)+\mu\pi(j+1,0)$$
$$+\eta\pi(j,1),\ j=1,2,\ldots; \quad (10)$$

$$(\lambda+\eta)\pi(0,1)=\gamma\{\pi(0,0)+\pi(1,0)\}; \quad (11)$$

$$(\lambda+\eta)\pi(j,1)=\lambda\pi(j-1,1)$$
$$+\gamma\pi(j+1,0),\ j=1,2,\ldots. \quad (12)$$

Define the partial probability generating functions as follows:

$$\Pi_0(s)=\sum_{j=0}^{\infty}\pi(j,0)s^j,\Pi_1(s)=\sum_{j=0}^{\infty}\pi(j,1)s^j. \quad (13)$$

By using (9) and (10), we get

$$\{\lambda s^2-(\lambda+\mu+\gamma)s+\mu\}\Pi_0(s)+\eta\, s\Pi_1(s)$$
$$=\mu(1-s)\pi(0,0) \quad (14)$$

By using (11) and (12), we get

$$\gamma\Pi_0(s)+s[\lambda s-(\lambda+\eta)]\Pi_1(s)=\gamma(1-s)\pi(0,0). \quad (15)$$

Solving (14) and (15), we get

$$[\Pi_0(s)/\Delta_1(s)]=[\Pi_1(s)/\Delta_2(s)]=[1/\Delta_3(s)], \quad (16)$$

where

$$\Delta_1(s)=s(1-s)\pi(0,0)[\mu\{\lambda s-(\lambda+\eta)\}-\gamma\eta],$$

$$\Delta_2(s)=\gamma s(1-s)\pi(0,0)[\lambda s-(\lambda+\mu+\gamma)],$$

$$\Delta_3=$$
$$s(s-1)\{\lambda^2s^2-\lambda s(\lambda+\mu+\gamma+\eta)+(\lambda\mu+\mu\eta+\eta\gamma)\}.$$

Consequently, we get by using (16),

$$\Pi_0(s)=\frac{\pi(0,0)[\gamma\eta-\mu\{\lambda s-(\lambda+\eta)\}]}{\lambda^2s^2-\lambda s(\lambda+\mu+\gamma+\eta)+(\lambda\mu+\mu\eta+\gamma\eta)}, \quad (17)$$

$$\Pi_1(s)=\frac{\gamma\pi(0,0)[(\lambda+\mu+\gamma)-\lambda s]}{\lambda^2s^2-\lambda s(\lambda+\mu+\gamma+\eta)+(\lambda\mu+\mu\eta+\gamma\eta)}. \quad (18)$$

Invoking the law of total probability, that is

$$\Pi_0(s)+\Pi_1(s)=1,$$

We get

$$\pi(0,0)=\frac{\eta(\mu+\gamma)-\lambda(\eta+\gamma)}{(\mu+\gamma)(\eta+\gamma)}$$
$$=1-\left[\frac{\gamma}{(\eta+\gamma)}+\frac{\lambda}{(\mu+\gamma)}\right]. \quad (19)$$

Since $0<\pi(0,0)<1$, we get the stability condition that

$$\lambda(\gamma+\eta)<\eta(\mu+\gamma). \quad (20)$$

The other steady-state probabilities are found from (17) and (18). To this end, we consider the zeros of the denominator in (17) and (18). The zeros are given by

$$\omega_1=\frac{(\lambda+\mu+\gamma+\eta)}{2\lambda}-\frac{\sqrt{(\lambda+\mu+\gamma+\eta)^2-4(\lambda\mu+\mu\eta+\eta\gamma)}}{2\lambda}, \quad (21)$$

$$\omega_2=\frac{(\lambda+\mu+\gamma+\eta)}{2\lambda}+\frac{\sqrt{(\lambda+\mu+\gamma+\eta)^2-4(\lambda\mu+\mu\eta+\eta\gamma)}}{2\lambda}. \quad (22)$$

We find that

$$\omega_1+\omega_2=\frac{\lambda+\mu+\gamma+\eta}{\lambda},\omega_1\omega_2=\frac{\lambda\mu+\mu\eta+\eta\gamma}{\lambda^2}. \quad (23)$$

Further, we have

$$(\lambda+\mu+\gamma+\eta)^2-4(\lambda\mu+\mu\eta+\eta\gamma)$$
$$=(\lambda-\mu-\gamma+\eta)^2+4\lambda\gamma>0.$$

So, ω_1 and ω_2 are positive and distinct. Using (20) in (23), we get $\omega_1+\omega_2>2,\ (\omega_1-1)(\omega_2-1)>0.$

Implies $\omega_1>1$ and $\omega_2>1$.

Then, by (17), we get

$$\Pi_0(s) = \frac{\pi(0,0)\left[\gamma\eta - \mu\{\lambda s - (\lambda+\eta)\}\right]}{\lambda^2(\omega_1 - s)(\omega_2 - s)}$$

$$= \frac{\pi(0,0)}{\lambda^2(\omega_2-\omega_1)}\left[(\lambda\mu+\mu\eta+\eta\gamma)\sum_{n=0}^{\infty}\left(\frac{1}{\omega_1^{n+1}}-\frac{1}{\omega_2^{n+1}}\right)s^n\right]$$
$$-\frac{\pi(0,0)}{\lambda^2(\omega_2-\omega_1)}\left[\lambda\mu\sum_{n=0}^{\infty}\left(\frac{1}{\omega_1^{n}}-\frac{1}{\omega_2^{n}}\right)s^n\right]. \tag{24}$$

Equating the coefficients of s^n in (24), we obtain

$$\pi(n,0) = \frac{\pi(0,0)}{\lambda^2(\omega_2-\omega_1)}\left[(\lambda\mu+\mu\eta+\eta\gamma)\left(\frac{1}{\omega_1^{n+1}}-\frac{1}{\omega_2^{n+1}}\right)\right] - \frac{\pi(0,0)}{\lambda^2(\omega_2-\omega_1)}\left[\lambda\mu\left(\frac{1}{\omega_1^{n}}-\frac{1}{\omega_2^{n}}\right)\right], n = 0,1,2,\ldots . \tag{25}$$

Similarly, by using (18), we get

$$\Pi_1(s) = \frac{\gamma\pi(0,0)\left[(\lambda+\mu+\gamma)-\lambda s\right]}{\lambda^2(\omega_1 - s)(\omega_2 - s)}$$

$$= \frac{\gamma\pi(0,0)}{\lambda^2(\omega_2-\omega_1)}\left[(\lambda+\mu+\gamma)\sum_{n=0}^{\infty}\left(\frac{1}{\omega_1^{n+1}}-\frac{1}{\omega_2^{n+1}}\right)s^n\right]$$
$$-\frac{\gamma\pi(0,0)}{\lambda^2(\omega_2-\omega_1)}\left[\lambda\sum_{n=0}^{\infty}\left(\frac{1}{\omega_1^{n}}-\frac{1}{\omega_2^{n}}\right)s^n\right]. \tag{26}$$

Equating the coefficients of s^n in (26), we obtain

$$\pi(n,1) = \frac{\gamma\pi(0,0)}{\lambda^2(\omega_2-\omega_1)}\left[(\lambda+\mu+\gamma)\left(\frac{1}{\omega_1^{n+1}}-\frac{1}{\omega_2^{n+1}}\right)\right] - \frac{\gamma\pi(0,0)}{\lambda^2(\omega_2-\omega_1)}\left[\lambda\left(\frac{1}{\omega_1^{n}}-\frac{1}{\omega_2^{n}}\right)\right], n = 0,1,2,\ldots . \tag{27}$$

We have obtained the following theorem by consolidating the above results:

Theorem: Under the condition $\lambda(\gamma+\eta) < \eta(\mu+\gamma)$, the steady-state probabilities of the states of the attack system are given by

$$\pi(n,0) = \frac{\pi(0,0)}{\lambda^2(\omega_2-\omega_1)}\left[(\lambda\mu+\mu\eta+\eta\gamma)\left(\frac{1}{\omega_1^{n+1}}-\frac{1}{\omega_2^{n+1}}\right)\right] - \frac{\pi(0,0)}{\lambda^2(\omega_2-\omega_1)}\left[\lambda\mu\left(\frac{1}{\omega_1^{n}}-\frac{1}{\omega_2^{n}}\right)\right], n = 1,2,\ldots .$$

$$\pi(n,1) = \frac{\gamma\pi(0,0)}{\lambda^2(\omega_2-\omega_1)}\left[(\lambda+\mu+\gamma)\left(\frac{1}{\omega_1^{n+1}}-\frac{1}{\omega_2^{n+1}}\right)\right] - \frac{\gamma\pi(0,0)}{\lambda^2(\omega_2-\omega_1)}\left[\lambda\left(\frac{1}{\omega_1^{n}}-\frac{1}{\omega_2^{n}}\right)\right], n = 0,1,2,\ldots ,$$

where

$$\pi(0,0) = 1 - \frac{\gamma(\mu+\gamma)+\lambda(\gamma+\eta)}{(\mu+\gamma)(\eta+\gamma)}.$$

IV. SYSTEM'S PERFORMANCE MEASURES

A. Idle state probability of the server

The probability that the server is in idle state not in repair, is given by

$$\pi(0,0) = 1 - \frac{\gamma(\mu+\gamma)+\lambda(\gamma+\eta)}{(\mu+\gamma)(\eta+\gamma)}. \tag{28}$$

For the purpose of illustration, let us fix the parameters $\lambda = 5.0$, $\mu = 6.0$, and $\eta = 5.0$. Then, the stability condition $\lambda(\gamma+\eta) < \eta(\mu+\gamma)$ is satisfied for all positive values of γ. Varying γ from 0.1 to 10.0, we obtain the corresponding values of $\pi(0,0)$ and present them in the following table:

Table- I: $\pi(0,0)$ for Varying γ from 0.1 to 10.0

γ	1	2	3	4	5
$\pi(0,0)$	0.244	0.167	0.117	0.083	0.060
γ	6	7	8	9	10

$\pi(0,0)$	0.043	0.030	0.021	0.014	0.008

We find that the probability $\pi(0,0)$ decreases as γ increases.

B. Busy state probability of the server

When the server is not under repair, to make the server busy, there must be at least one customer in the system. Let α be the probability that the server is in busy state. Then, by using (24), we obtain

$$\alpha=\sum_{n=1}^{\infty}\pi(n,0)=\Pi_0(1)-\pi(0,0)$$

$$=\frac{\pi(0,0)\left[\gamma\eta-\mu\{\lambda-(\lambda+\eta)\}\right]}{\lambda^2(\omega_1-1)(\omega_2-1)}-\pi(0,0)=\frac{\lambda}{(\mu+\gamma)} \tag{29}$$

Clearly, α always decreases as γ increases.

C. Server repair time probability

Let β be the probability that the server is undergone repair.

Then, by using (26), we obtain

$$\beta=\sum_{n=0}^{\infty}\pi(n,1)=\prod\nolimits_1(1)$$

$$=\frac{\gamma\pi(0,0)\left[(\lambda+\mu+\gamma)-\lambda\right]}{\lambda^2(\omega_1-1)(\omega_2-1)}$$

$$=\frac{\gamma\pi(0,0)(\mu+\gamma)}{\lambda^2(\omega_1-1)(\omega_2-1)}=\frac{\gamma}{\eta+\gamma} \tag{30}$$

Clearly β always decreases as η increases.

D. Average number of customers of the attack system in the long run

Let ϕ be the mean number of customers in the system. Then, after simple algebraic simplifications, we obtain,

$$E(X)=\phi=\left[\frac{d}{ds}\{\Pi_0(s)+\Pi_1(s)\}\right]_{s=1}$$

$$=\frac{\lambda}{(\eta+\gamma)}\left[\frac{\gamma(\mu+\gamma)+\eta(\eta+\gamma))}{\eta(\mu+\gamma)-\lambda(\gamma+\eta)}\right] \tag{31}$$

Taking $\lambda=5.0,\mu=8.0,\eta=4.0,$ and varying γ from 1.0 to 10.0, we get the following table:

In this, we find that the size of the system increases as γ increases and is depicted in Table-II and Fig. 2.

Table- II: $E(X)$ by Varying γ from 0.1 to 10.0 and fixing $\lambda=5.0,\mu=8.0,\eta=4.0$

γ	1.0	2.0	3.0	4.0	5.0
$E(X)$	2.636	3.667	4.841	6.25	8.016
γ	6.0	7.0	8.0	9.0	10.0
$E(X)$	10.333	13.546	18.333	26.282	42.143

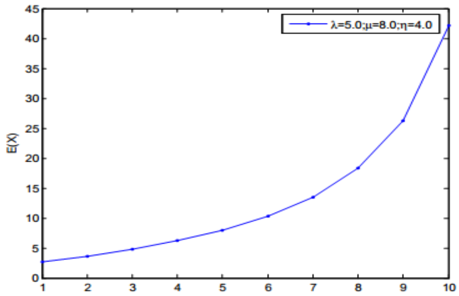

Fig. 2. Attack system's average customers by fixing $\lambda=5.0,\mu=8.0,\eta=4.0$

We find that the size of the system increases as γ increases and is depicted in Table-II and Fig. 1.

Table- III: $E(X)$ by Varying γ from 0.1 to 10.0 and fixing $\lambda=5.0,\mu=6.0,\eta=7.0$

γ	1.0	2.0	3.0	4.0	5.0
$E(X)$	4.375	3.990	3.731	3.546	3.407
γ	6.0	7.0	8.0	9.0	10.0
$E(X)$	3.300	3.214	3.145	3.086	3.039

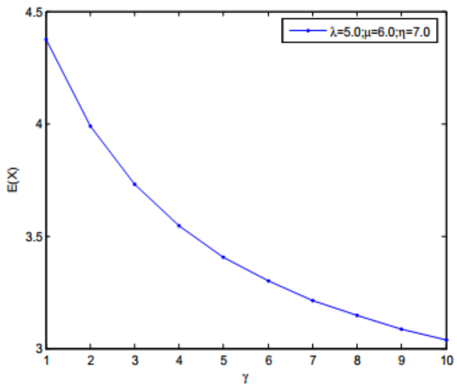

Fig. 3. Attack system's average customers by fixing

$$\lambda = 5.0, \mu = 6.0, \eta = 7.0$$

In this case, we observed that the size of the system decreases as γ increases and the situation is depicted in Table-III and Fig. 3.

V. CONCLUSION

In the present paper, a new queue model for studying a communication system where the server fails due to the occurrence of attacks, washing out only the customer undergoing service, but not the waiting customers. A repair facility is considered for repairing the failed server, and customers are allowed to join the queue during the repair time. Explicit expressions are provided for the stationary probabilities of the attack system. A numerical illustration has been provided which has validated the effects of attacks on the performance of the model.

REFERENCES

1. O. Boudali and A. Economou, "Optimal and equilibrium balking strategies in the single server Markovian queue with catastrophes," European Journal of Operational Research, vol. 218, 2012, pp. 708-715.
2. O. Boudali and A. Economou, "The effect of Catastrophes on the Strategic Customer Behavior in Queueing Systems," Naval Research Logistics, vol. 60, 2013, pp. 571-587.
3. B. Krishna Kumar and S. PavaiMadheswari, " Transient Analysis of an $M/M/1$ Queue Subject to Catastrophes and Server Failures," Stochastic Analysis and Applications, vol. 23, 2005, pp. 329-340.
4. G. C. Mytalas and M. A. Zazanis, "An $M_X/G/1$ queueing system with disasters and repairs under a multiple adapted vacation policy," Naval Research Logistics, vol. 62, 2015, pp. 171-189.
5. S.K. Srinivasan, Stochastic Point Processes and Their Applications, Methuen, 1974.
6. M. F. Yassen and A. M. K. Tarabia, "Transient Analysis of MarkovianQueueing" System with Balking and Reneging Subject to Catastrophes and Server Failures, Appl. Math. Inf.Sci., vol. 11, 2017, pp. 1041-1047

AUTHORS PROFILE

Jagadeesan Viswanath received his M.Sc. and M.Phil.degrees from University of Madras, Tamilnadu, India and PhD degree in Mathematics at Vel Tech Rangarajan Dr.Sagunthala R&D Institute of Science and Technology, Avadi, Chennai, India. He is presently working as Associate Professor of Mathematics in Department of Mathematics, Vel TechRangarajan Dr. Sagunthala R&D Institute of Science and Technology, Avadi, Chennai, India. His research interests are in the areas of applied Mathematics and Stochastic modelling of dynamical systems. He has published research articles in reputed international journals of mathematical and engineering sciences.

Swaminathan Udayabaskaran received his MSc and PhD from Indian Institute of Technology, Madras, Chennai, India. He is at present working as Professor of Mathematics in VelTech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology, Avadi, Chennai, India. Earlier he worked as Reader in Mathematics at Presidency College, Chennai600005, Tamilnadu, India. His research interest includes stochastic analysis of queueing systems, inventory systems and biological systems.