Double Arithmetic Odd Decomposition [DAOD] of Some Complete 4-Partite Graphs

V. G. Smilin Shali, S. Asha

***Abstract:** Let G be a finite, connected, undirected graph without loops or multiple edges. If $G_1, G_2, \dots, G_n$ are connected edge – disjoint subgraphs of G with $E(G) = E(G_1) \cup E(G_2) \cup \dots \cup E(G_n)$, then $\{G_1, G_2, \dots, G_n\}$ is said to be a decomposition of G. The concept of Arithmetic Odd Decomposition [AOD] was introduced by E. Ebin Raja Merly and N. Gnanadhas. A decomposition $\{G_1, G_2, \dots, G_n\}$ G is said to be Arithmetic Decomposition if each G_i is connected and $|E(G_i)| = a + (i-1)d$, for $1 \le i \le n$ and $a, d \in \mathbb{Z}$. When a =1 and d = 2, we call the Arithmetic Decomposition as Arithmetic Odd Decomposition. A decomposition $\{G_1, G_3, \dots, G_{2n-1}\}$ of G is said to be AOD if $|E(G_i)| = i$, $\forall\ i = 1, 3, \dots, 2n-1$. In this paper, we introduce a new concept called Double Arithmetic Odd Decomposition [DAOD]. A graph G is said to have Double Arithmetic Odd Decomposition [DAOD] if G can be decomposed into 2k subgraphs $\{2G_1, 2G_3, \dots, 2G_{2k-1}\}$ such that each G_i is connected and $|E(G_i)| = i$, $\forall\ i = 1, 3, \dots, 2k-1$. Also we investigate DAOD of some complete 4-partite graphs such as $K_{2,2,2,m}$, $K_{2,4,4,m}$ and $K_{1,2,4,m}$.*

***Keywords:** Decomposition of graph, Arithmetic Decomposition, Arithmetic Odd Decomposition [AOD], Double Arithmetic Odd Decomposition [DAOD].*

I. INTRODUCTION

Let G = (V,E) be a simple, connected graph with p vertices and q edges. If $G_1, G_2, \dots, G_n$ are connected edge – disjoint subgraphs of G with $E(G) = E(G_1) \cup E(G_2) \cup \dots \cup E(G_n)$, then $\{G_1, G_2, \dots, G_n\}$ is said to be a Decomposition of G. The concept of Arithmetic ODD Decomposition [AOD] was introduced by E. Ebin Raja Merly and N. Gnanadhas [1]. In this paper, we introduce a new concept called Double Arithmetic Odd Decomposition [DAOD]. Terms not defined here are used in the sense of Harary [3].

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II. PRELIMINARIES

Definition 2.1. Let G = (V, E) be a simple graph of order p and size q. If $G_1, G_2, \dots, G_n$ are edge - disjoint subgraphs of G such that $E(G) = E(G_1) \cup E(G_2) \cup \dots \cup E(G_n)$, then $\{G_1, G_2, \dots, G_n\}$ is said to be a Decomposition of G.

Definition 2.2. A decomposition $\{G_1, G_2, \dots, G_n\}$ of a connected graph G is said to be a Linear Decomposition or Arithmetic Decomposition if each G_i is connected and $|E(G_i)| = a + (i-1)d$, for $1 \le i \le n$ and $a, d \in \mathbb{Z}$. Clearly $q = \frac{n}{2}[2a + (n-1)d]$.

Definition 2.3. When a = 1 and d =1, the size of G is $q = \frac{n(n+1)}{2}$. When a =1 and d = 2, the size of G is $q = n^2$. Hence the number of edges of G is a perfect square. Since the number of edges of G is a perfect square, q is the sum of first n odd numbers 1, 3, 5, . . . ,(2n -1). Thus we call the Arithmetic Decomposition with a = 1 and d = 2 as Arithmetic Odd Decomposition [AOD]. Since the number of edges of each subgraph of G is odd, we denote the AOD as $\{G_1, G_3, \dots, G_{2n-1}\}$.

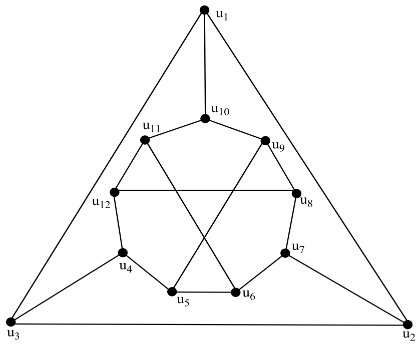
Theorem 2.4. Any connected graph G admits AOD $\{G_1, G_3, \dots, G_{2n-1}\}$ where $G_i = (V_i, E_i)$ and $|E(G_i)| = i$, for $i = 1, 3, \dots, 2n-1$ if and only if $q = n^2$, for every $n \in \mathbb{Z}^+$.

Definition 2.5. A Complete 4-Partite Graph is a 4-partite graph whose vertices are decomposed in to 4 disjoint sets such that no two vertices with in the same set are adjacent but every pair of vertices in the 4 sets are adjacent. A complete 4-partite graph is denoted as $K_{a,b,c,d}$ where a, b, c and d are 4 disjoint set of vertices of the graph.

III. DOUBLE ARITHMETIC ODD DECOMPOSITION [DAOD] OF GRAPHS

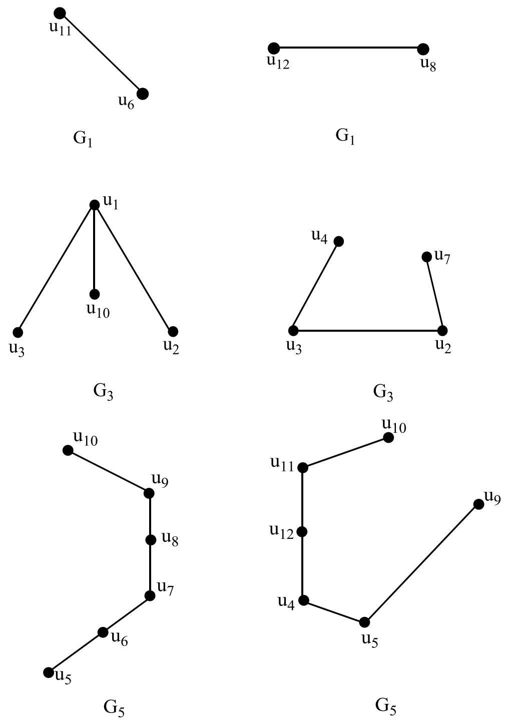
Definition 3.1. A graph G is said to have Double Arithmetic Odd Decomposition [DAOD] if G can be decomposed in to 2k subgraphs $\{2G_1, 2G_3, \dots, 2G_{2k-1}\}$ such that each G_i is connected and $|E(G_i)| = i$, $\forall\ i = 1, 3, 5, \dots, 2k-1$. Clearly $q = 2k^2$. We denote DAOD as $\{2G_1, 2G_3, \dots, 2G_{2k-1}\}$.

Example 3.2. Consider the following Tietze's graph G.



Tietze's graph G

Tietze's graph G admits DAOD. The DAOD of G is as follows.



IV. DAOD OF K2,2,2,M

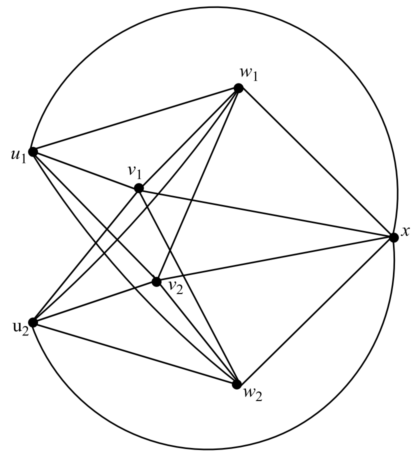
Theorem 4.2. For any integer m, $K_{2,2,2,m}$ has a DAOD $\{2G_1, 2G_3, \ldots, 2G_{2k-1}\}$ [2k- decompositions] if and only if there exists an integer k satisfying the following properties.

1. $k = 3r$, $r \geq 1$ and $r \in Z$
2. $k^2 = 6 + 3m$

Proof . Let $G = K_{2,2,2\,m}$. Assume that G has a DAOD $\{2G_1, 2G_3, \ldots, 2G_{2k-1}\}$. By the definition of DAOD, $q = 2k^2$, where k denotes the total number of decompositions. Clearly, there will be 2 copies of k decompositions . By the definition of $K_{2,2,2\,m}$, $q = 12 + 6m$. Hence $2k^2 = 12+6m$. Thus $k^2 = 6 + 3m$. This is possible only when k is a multiple of 3. Suppose $k = 3r$, $r \geq 1$ and $r \in Z$. Then $(3r)^2 = 6 + 3m$. This implies $m = 3r^2-2$, an integer. Hence $k = 3r$, $r \geq 1$ and $r \in Z$.

Conversely, suppose $k = 3r$, $r \geq 1$ and $r \in Z$ and $k^2 = 6 + 3m$. Let $G = K_{2,2,2\,m}$. By the definition of G, $q = 12 + 6m = 2k^2$. Since $q = 2k^2$, G can be decomposed in to $\{2G_1, 2G_3, \ldots, 2G_{2k-1}\}$. Hence G admits DAOD.

Illustration 4.2. As an illustration, let us decompose $K_{2,2,2,1}$.



$K_{2,2,2,1}$

Let $G = K_{2,2,2,1}$. G admits DAOD $\{2G_1, 2G_3, \ldots, 2G_5\}$as follows.

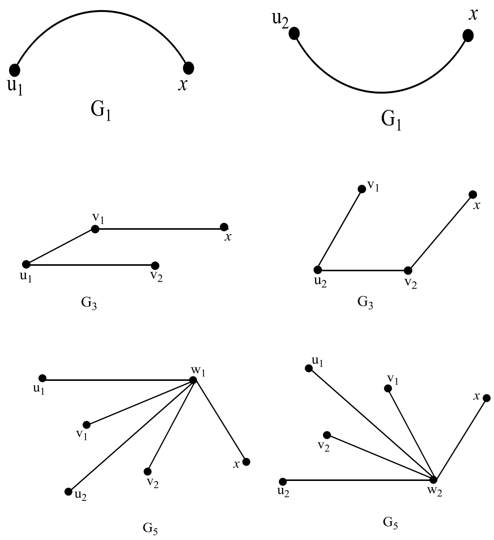


Table 4.3. List of first 10 $K_{2,2,2,m}$'s that admits DAOD and their decompositions are listed below.

m	DAOD
1	$2G_1, 2G_3, 2G_5$
10	$2G_1, 2G_3, \ldots, 2G_{11}$
25	$2G_1, 2G_3, \ldots, 2G_{17}$
46	$2G_1, 2G_3, \ldots, 2G_{23}$
73	$2G_1, 2G_3, \ldots, 2G_{29}$
106	$2G_1, 2G_3, \ldots, 2G_{35}$
145	$2G_1, 2G_3, \ldots, 2G_{41}$
190	$2G_1, 2G_3, \ldots, 2G_{47}$
241	$2G_1, 2G_3, \ldots, 2G_{53}$
298	$2G_1, 2G_3, \ldots, 2G_{59}$

V. DAOD OF K2,4,4,M

Lemma 5.1 . Let $k^2 = 16 + 5m$. If $k–6 \equiv 0 (\text{mod } 10)$, then $2k^2$ can be decomposed in to $\{2G_1, 2G_3, . . ., 2G_{2k-1}\}$.

Proof . We have $k– 6 \equiv 0$ (mod 10). Then $k- 6 = 10r$, $r = 0,1, 2, \ldots$. Hence $k = 6 + 10r$. Proof is by induction on r. When $r = 0$, $k = 6$. Then $2k^2 = 72$ can be decomposed in to $\{2G_1, 2G_3, . . . , 2G_{11}\}$. Hence the result is true for $r = 0$.

Assume that the result is true for $r – 1$. Then $k = 10(r–1) + 6 = 10r – 4$. Now, $2k^2 = 2[10r – 4]^2 = 200r^2 – 160r + 32$. Thus $2k^2$ can be decomposed in to $\{2G_1, . . . , 2G_{2r-3}\}$.

Now to prove that the result is true for r. Let $k = 10r +6$. Then $2k^2 = 200r^2 + 240r + 72$ we have to prove that $200r^2 + 240 r + 72$ can be decomposed in to $\{2G_1, 2G_3, . . ., 2G_{2r-1}\}$. Define $2k^2 = 2[10r –4]^2 \cup \{20r – 7 , 20r – 5 , . . . , 20r + 11 \}$. Then by the induction hypothesis , we get $200r^2 – 160r + 32+ 2[20r -7 + 20r -5 + . . . + 20r + 9+20r +11]$ can be decomposed in to $\{2G_1, 2G_{2r-3}, . . . , 2G_{2r-1}\}$. That is $200r^2 – 160r + 32 + 2[200r + 20]$ can be decomposed in to $\{2G_1, . . . , 2G_{2r-1}\}$. Thus $200r^2 + 240 r + 72$ can be decomposed in to $\{2G_1, 2G_3, . . . , 2G_{2r-1}\}$. Hence by induction hypothesis, if $k– 6 \equiv 0 (\text{mod } 10)$, then $2k^2$ can be decomposed in to $\{2G_1, . . . , 2G_{2k-1}\}$. This completes the proof.

Lemma 5.2. Let $k^2 = 16+5m$. If $k–9 \equiv 0 (\text{mod } 10)$, then $2k^2$ can be decomposed in to $\{2G_1, 2G_3, . . ., 2G_{2k-1}\}$.

Proof . We have $k– 9 \equiv 0$(mod 10). Then $k- 9 = 10r$, $r = 0, 1, 2, \ldots$. Hence $k = 9 +10r$. Proof is by induction on r. When $r = 0$, $k = 9$. Then $2k^2 = 162$ can be decomposed into $\{2G_1, . . . , 2G_{17} \}$. Hence the result is true for $r = 0$.

Assume that the result is true for r-1. Then $k = 10(r-1) + 9 = 10r –1$. Now $2k^2 = 2[10r –1]^2 = 200r^2 – 40r + 2$. Thus $2k^2$ can be decomposed in to $\{2G_1, 2G_3, . . . , 2G_{2r-3} \}$.

Now, to prove that the result is true for r. Let $k = 9 + 10r$. Then $2k^2 = 200r^2 + 360r + 162$. We have to prove that $200r^2 + 360r +162$ can be decomposed in to $\{2G_1, 2G_3, . . . , 2G_{2r-1}\}$. Define $2k^2 = 2[10r –1]^2 \cup \{20r – 1 , 20r + 1 , . . . , 20r + 17 \}$. Then by the induction hypothesis , we get $200r^2 – 40 r + 2 + 2 [20r -1 + 20r + 1 + . . . + 20r + 17]$ can be decomposed in to $\{2G_1, . . . , 2G_{2r-1}\}$.That is $200r^2 – 40 r + 2 + 2 [200r + 80]$ can be decomposed in to $\{2G_1, 2G_3, . . . , 2G_{2r-1}\}$. Thus $200r^2 + 360r +162$ can be decomposed in to $\{2G_1, 2G_3, . . . , 2G_{2r-1}\}$. Hence by induction hypothesis , if $k – 9 \equiv 0 (\text{mod } 10)$, then $2k^2$ can be decomposed in to $\{2G_1, 2G_3, . . . , 2G_{2k-1}\}$.This completes the proof.

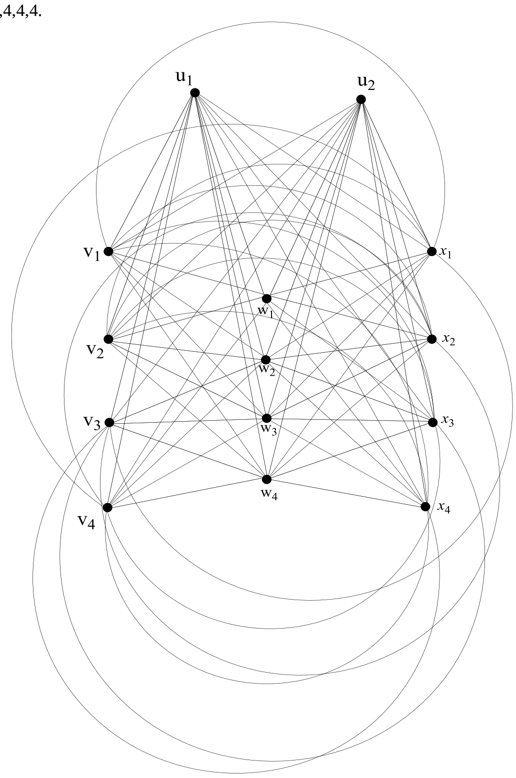
Theorem 5.3. For any integer m, $K_{2,4,4,m}$ has a DAOD $\{2G_1, 2G_3, . . . , 2G_{2k-1}\}$ [2k – decompositions] if and only if there exists an integer k satisfying the following properties.

1. $k = 6 + 10r$ or $k = 9 + 10r$, $r = 0,1,2, \ldots$
2. $k^2 = 16 + 5 m$

Proof . Let $G = K_{2,4,4,m}$. Assume that G admits DAOD. By the definition of DAOD, $q = 2k^2$, where k denotes the total number of decompositions. Clearly, there will be 2 copies of k decompositions . By the definition of G, $q = 32+10 m$. Hence $2k^2 = 32 + 10 m$. Thus $k^2 = 16 + 5m$. Hence $m = \frac{k^2 -16}{5}$. Since m is an integer, either $k – 6 \equiv 0$(mod 10) or $k -9 \equiv 0$(mod 10). Hence $k = 6 + 10r$ or $9 + 10 r$, $r =0, 1, 2, \ldots$

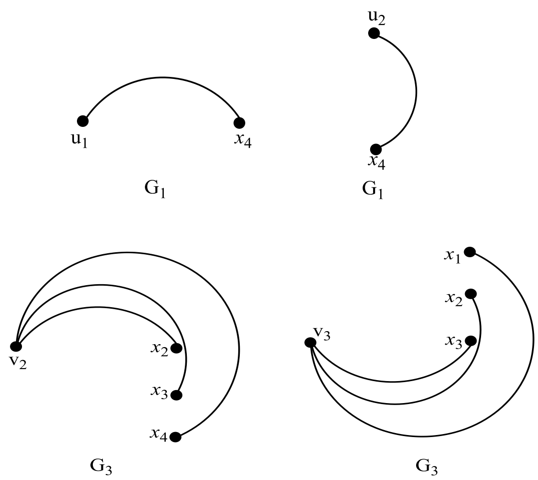
Conversely, assume that $k = 6 + 10r$ or $k = 9 + 10r$, $r = 0, 1, 2, \ldots$ and $k^2 = 16 + 5 m$. Let $G = K_{2,4,4,m}$. By Lemma 5.1 and 5.2, $2k^2$ can be decomposed in to $\{2G_1, . . . , 2G_{2k-1} \}$. Thus G admits DAOD.

Illustration 5.4. As an illustration, let us decompose $K_{2,4,4,4}$.



$K_{2,4,4,4}$

Let $G = K_{2,4,4,4}$. G admits DAOD $\{ 2G_1, 2G_3, 2G_5, 2G_7, 2G_9, 2G_{11}\}$ as follows



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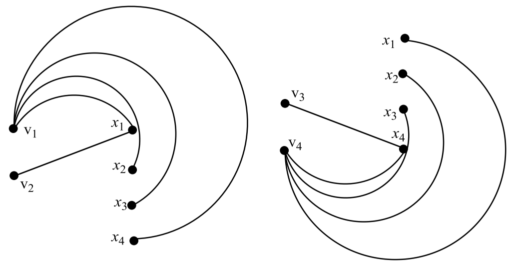
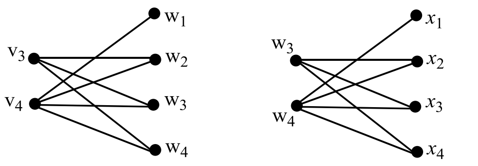
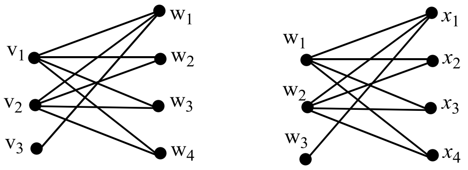


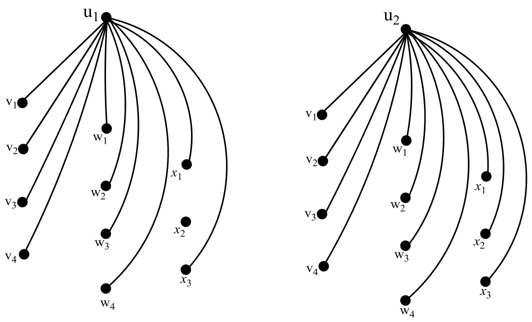
G_5 G_5



G_7 G_7



G_9 G_9



G_{11} G_{11}

Table 5.5. List of first 10 $K_{2,4,4,m}$'s which accepts DAOD and their decompositions are given below.

m	DAOD
4	$2G_1, 2G_3, \ldots, 2G_{11}$
13	$2G_1, 2G_3, \ldots, 2G_{17}$
48	$2G_1, 2G_3, \ldots, 2G_{31}$
69	$2G_1, 2G_3, \ldots, 2G_{37}$
132	$2G_1, 2G_3, \ldots, 2G_{51}$
165	$2G_1, 2G_3, \ldots, 2G_{57}$
256	$2G_1, 2G_3, \ldots, 2G_{71}$
301	$2G_1, 2G_3, \ldots, 2G_{77}$
420	$2G_1, 2G_3, \ldots, 2G_{91}$
477	$2G_1, 2G_3, \ldots, 2G_{97}$

VI. DAOD OF K1,2,4,M

Theorem 6.1. For an even integer m, $K_{1,2,4,m}$ has a DAOD $\{2G_1, 2G_3, \ldots, 2G_{2k-1}\}$ [2k- decompositions] if and only if there exists an integer k satisfying the following properties.

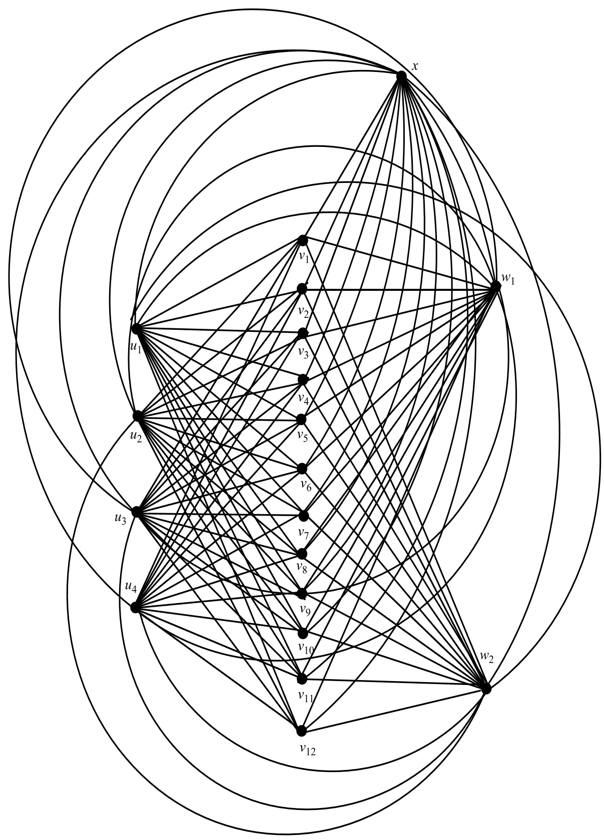
1. $k = 7r$, $r \geq 1$ and $r \in Z$
2. $2k^2 = 14 + 7m$

Proof . Let $G = K_{1,2,4,m}$. Assume that G admits DAOD. By the definition of DAOD, $q = 2k^2$, where k denotes the total number of decompositions. Clearly, there will be two copies of k decompositions . By the definition of G, $q = 14 + 7m$. Hence $2k^2 = 14 + 7$. Clearly $14 + 7m$ is even. Hence m is even.

Also $m = \dfrac{2k^2 - 14}{7}$. Since m is an even integer, $2k^2 - 14$ should be a multiple of 7. This is possible only when $k = 7r$, $r \geq 1$ and $r \in Z$. Suppose $k = 7r$. Then $2(49r^2) = 14 + 7m$ and hence $m = 14r^2 - 2$, an integer . Thus k satisfies the two properties.

Conversely, assume that $k = 7r$, $r \geq 1$ and $r \in Z$ and $2k^2 = 14 + 7m$. Let $G = K_{1,2,4,m}$. Then $q = 14 + 7m = 2k^2$. Since $q = 2k^2$, G can be decomposed in to $\{2G_1, 2G_3, \ldots, 2G_{2k-1}\}$. Hence G admits DAOD.

Illustration 6.2. As an illustration , let us decompose $K_{1,2,4,12}$



$K_{1,2,4,12}$

Let $G = K_{1,2,4,12}$. G admits DAOD and its decompositions are given below

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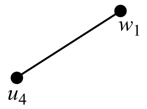
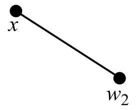
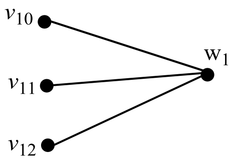


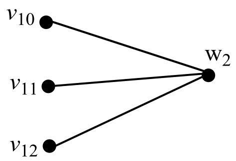
G_1



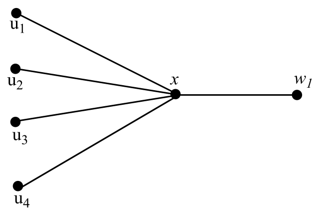
G_1



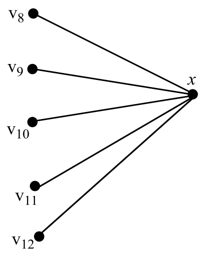
G_3



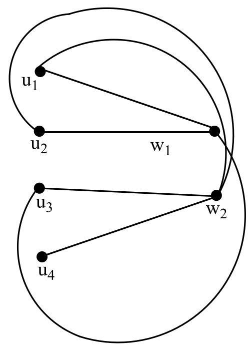
G_3



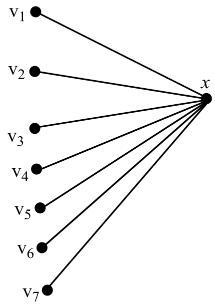
G_5



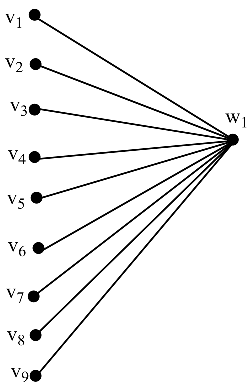
G_5



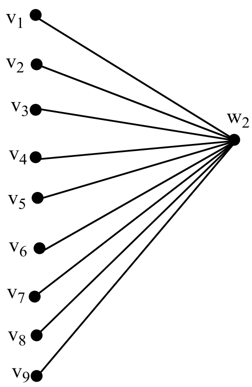
G_7



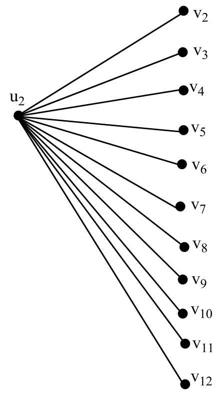
G_7



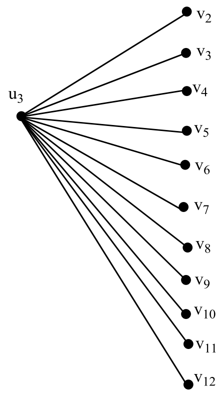
G_9



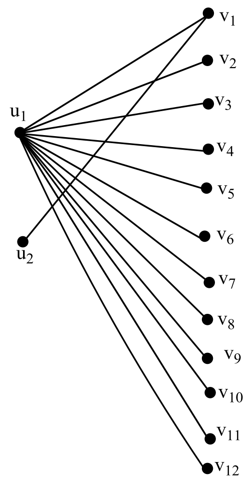
G_9



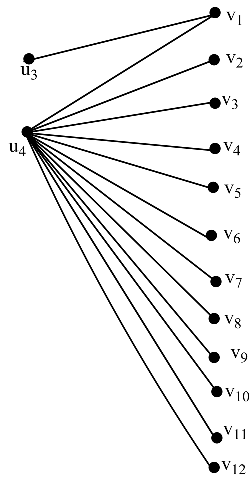
G_{11}



G_{11}



G_{13}



G_{13}

Table 6.3. List of first 10 $K_{1,2,4,m}$'s which accepts DAOD and their decompositions are given below.

m	DAOD
12	$2G_1, 2G_3, \ldots, 2G_{13}$
54	$2G_1, 2G_3, \ldots, 2G_{27}$
124	$2G_1, 2G_3, \ldots, 2G_{41}$
222	$2G_1, 2G_3, \ldots, 2G_{55}$
348	$2G_1, 2G_3, \ldots, 2G_{69}$
502	$2G_1, 2G_3, \ldots, 2G_{83}$
684	$2G_1, 2G_3, \ldots, 2G_{97}$
894	$2G_1, 2G_3, \ldots, 2G_{111}$
1132	$2G_1, 2G_3, \ldots, 2G_{125}$
1398	$2G_1, 2G_3, \ldots, 2G_{139}$

VII. CONCLUSION

Thus we can extend this Double Arithmetic Odd Decomposition for various 4-partite graphs. This decomposition technique plays a major role in the area of decomposition.

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