

# Vertex Antimagic Edge Labeling of Cube of A Path Graph



R. Sreenivasan, M.S. Paulraj

**Abstract:** *There are many concepts in science that are very hard to understand and to make use of them in a effective way, it is atmost important to have a tool that best explains these complex concepts in a simpler way. Graph theory is one of the most interesting topics in mathematics that was used to explain many complicated concepts in a simpler and easier way. Graph theory is not just about points and lines and above all, there are many interesting topics in graph theory which has motivated many scholars to pursue research in different areas. One of the most interesting and elite topics in graph theory is the path. The researchers have discovered different types of concepts using paths and have proved different characteristics. Cube of a path graphs are one of those fascinating graphs that have evolved from paths and has been proved to admit a variety of properties. Like paths, labeling is also an area where graph theoretic researchers have shown great interest and have come up with different types of labeling. With the discovery of a spate of labeling, it has motivated and kindled the researchers to apply these labeling to a variety of graphs and check the admittance of different types of properties. One such intriguing type of labeling is the vertex antimagic edge labeling. In this paper, we will show that the cube of a path graph admits vertex antimagic edge labeling.*

**Keywords:** *Cube Of A Path Graph, Vertex Antimagic Edge Labelling* **AMS Subject Classification:** 05C78

## I. INTRODUCTION

There are many abstract things in nature that are very hard to comprehend and fathom. To understand these things, it is imperative to have a tool that can represent a complex entity in a simpler form. Thus, the need for the graphs became inevitable, which are used to explain the relationship between different persons and attributes that are considered only imaginable. The graph theoretic researchers have introduced a variety of graphs and one such graph is the cube of a path graph. With variety of applications inherited in itself, the paths have been one of the much sought after research topics in graph theory. The motivation behind the development of this paper is the definition of the cube of a path and its applications.

## II. VERTEX ANTIMAGIC EDGE LABELING

One of the most intense and profound topics in graph theory is labeling.

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In the recent years, many graph theoretic scholars and researchers have discovered many types of labeling. One of the eloquent types in labeling is the magic and antimagic labeling. The term labeling refers to the weight assigned to a vertex or an edge. Celebrated graph theory scholar J. Sedlacek [8] first introduced the thought of magic labeling.

The idea of antimagic labeling was introduced first by the illustrious duo of N. Hartsfield and J. Ringel [4]. They further proved that path  $P_n$ , cycles  $C_n$ , wheels  $W_n$  and regular graphs are all antimagic labelled graphs.

The notion of vertex antimagic edge labeling was first developed by two of the reputed and exalted scholars R. Bodendiek and G. Walther [1] and the same was compiled efficaciously by the renowned graph theorist J.A. Gallian [2]. The concept of vertex antimagic edge labeling was first defined by Martin Baca and Mirka Miller [5] as; A connected graph  $(V(G), E(G))$  is said to be an  $(a, d)$  antimagic edge labelled graph if there exists a positive integer  $a$  and a non-negative integer  $d$  and a bijection  $f: E \rightarrow \{1, 2, \dots, |E(G)|\}$  such that the induced mapping  $g: V(G) \rightarrow W$ , where  $W = \{a, a+d, a+2d, \dots, a+|V-1|d\}$  is also a bijection.

## III. CUBE OF A PATH GRAPH

The square of a path graph, as defined by G.H. Fan and H.A. Kierstead [3], is the graph obtained by joining every pair of vertices of distance two in the path. Similarly, the cube of a path graph is obtained by joining every pair of vertices of distance three in the path. R. Sreenivasan and M.S. Paulraj [9] used this definition effectively and have proved that the square of a path graph  $G(P_n^2)$  admits vertex antimagic edge labeling. The graph we consider here is a cube graph on path  $P_n$ , denoted by  $G(P_n^3)$  and we will prove that the cube of a path graph admits vertex antimagic edge labeling.

## IV. LABELING METHODOLOGY

The labeling of the edges is done in a manner such that labels are unique. To maintain the uniqueness, the edges of the graphs are labelled in a particular pattern. That is, for graphs on vertices  $n \geq 5$ ,  $n \neq 11 + 4k$ , and  $n \neq 9 + 4k$ ,  $k \in \mathbb{Z}^*$ , the edges are labelled following the definition of the function  $f^*: E(G) \rightarrow \mathbb{N}$  as;

$$f^*(u_i u_j) = i, j = i + 1, 1 \leq i \leq n - 1$$

$$f^*(u_i u_j) = i + (n - 1), j = i + 3, 1 \leq i \leq n - 3$$

For cube of path graphs drawn on vertices  $n = 11 + 4k$ , and  $n \neq 9 + 4k$ ,  $k \in \mathbb{Z}^*$ , the labeling of edges is done based on the function defined as

$$f^*(u_i u_j) = i, j = i + 3, 1 \leq i \leq n - 3$$

$$f^*(u_i u_j) = i + (n - 3), j = i + 1, 1 \leq i \leq n - 1$$

## Vertex Antimagic Edge Labeling of Cube of A Path Graph

For graphs on  $n = 9+4k$  vertices, the labels are not unique irrespective of the nature of labeling.

### V. MAIN RESULTS

**Theorem – 1:** The graph cube of a path  $P_n$  denoted by  $G(P_n^3)$ ,  $n \geq 5$ ,  $n \neq 11 + 4k$  and  $n \neq 9 + 4k$ ,  $k \in \mathbb{Z}^*$  admits vertex antimagic edge labeling.

*Proof:* Let  $P_n$  be a path on  $n$  vertices. Consider the cube of a path graph denoted  $G(P_n^3)$  that is derived from the path  $P_n$ . In this theorem, we give proof for cube graphs on path  $P_n$ ,  $n \geq 5$ ,  $n \neq 11 + 4k$  and  $n \neq 9 + 4k$ ,  $k \in \mathbb{Z}^*$ . To label the edges, define a function  $f^*: E(G) \rightarrow N$  as follows;

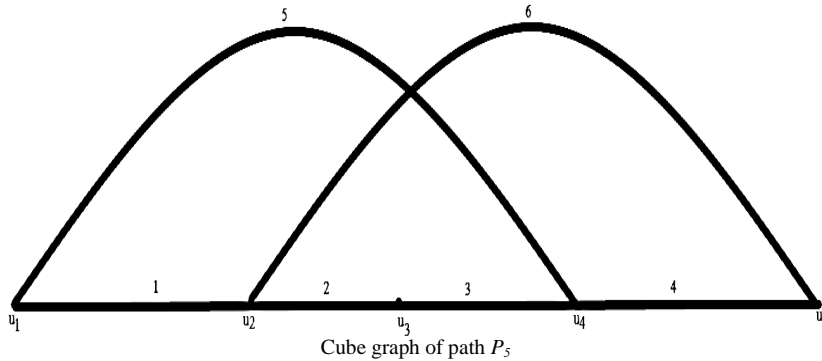
$$f^*(u_i u_j) = i, j = i + 1, 1 \leq i \leq n - 1$$

$$f^*(u_i u_j) = i + (n - 1), j = i + 3, 1 \leq i \leq n - 3$$

*Claim:* We claim that the edge labels are all distinct.

*Case – 1:* For some  $i \neq k$  in  $1 \leq i \leq n - 1$ , assume that  $f^*(u_i u_j) = f^*(u_k u_j)$

$$\Rightarrow i = k, \text{ a contradiction}$$

$$\Rightarrow \text{the edge labels are all distinct.}$$


In the above cube of path graph, the edge labels are defined based on the function  $f^*: E(G) \rightarrow N$  as

$$f^*(u_i u_j) = i, j = i + 1, 1 \leq i \leq 4$$

$$f^*(u_i u_j) = i + 4, j = i + 3, i = 1, 2$$

The above labeling pattern implies that the edge labels are distinct. The label of a vertex is the sum of labels of the edges that are incident to it. As the edge labels are distinct, the vertex labels are distinct too. Hence the graph  $G(P_5^3)$  admits vertex antimagic edge labeling.

**Theorem – 2:** The graph cube of a path  $P_n$  denoted by  $G(P_n^3)$ ,  $n = 11 + 4k$  and  $n \neq 9 + 4k$ ,  $k \in \mathbb{Z}^*$  admits vertex antimagic edge labeling.

*Proof:* Let  $P_n$  be a path on  $n$  vertices. Consider the cube of a path graph denoted by  $G(P_n^3)$ . In this theorem, we give proof for cube graphs on  $n = 11 + 4k$ ,  $k \in \mathbb{Z}^*$  vertices. To label the edges of the graph, define a function  $f^*: E(G) \rightarrow N$  as follows;

$$f^*(u_i u_j) = i, j = i + 3, 1 \leq i \leq n - 3$$

$$f^*(u_i u_j) = i + (n - 3), j = i + 1, 1 \leq i \leq n - 1$$

*Claim:* We claim that the edge labels are all distinct.

*Case – 1:* For some  $i \neq k$  in  $1 \leq i \leq n - 3$ , assume

$$\text{that } f^*(u_i u_j) = f^*(u_k u_j)$$

*Case – 2:* For some  $i \neq k$  in  $1 \leq i \leq n - 3$ , assume

$$\text{that } f^*(u_i u_j) = f^*(u_k u_j)$$

$$\Rightarrow i + n - 1 = k + n - 1$$

$$\Rightarrow i = k, \text{ a contradiction}$$

$$\Rightarrow \text{the edge labels are all distinct.}$$

*Case – 3:* For some  $i_1$  in  $1 \leq i \leq n - 1$  and  $i_2$  in

$$1 \leq i \leq n - 3, \text{ assume that } f^*(u_{i_1} u_j) = f^*(u_{i_2} u_j)$$

$$\Rightarrow i_1 = i_2 + n - 1,$$

$$\Rightarrow i_1 - i_2 = n - 1, \text{ a contradiction}$$

$$\Rightarrow \text{the edge labels are all distinct.}$$

From the claim discussed above, it has been proved that the edge labels of the graph are all distinct. The label of a vertex is defined as the sum of labels of the edges that are incident in it. Since the labels of the edges are all distinct, it follows that the vertex labels are also distinct. Hence the cube of path graph  $G(P_n^3)$ ,  $n \geq 5$ ,  $n \neq 11 + 4k$  and  $n \neq 9 + 4k$ ,  $k \in \mathbb{Z}^*$  admits vertex antimagic edge labeling. As an example, consider the graph of  $G(P_5^3)$  given by

$$\Rightarrow i = k, \text{ a contradiction}$$

$$\Rightarrow \text{the edge labels are all distinct.}$$

*Case – 2:* For some  $i \neq k$  in  $1 \leq i \leq n - 1$ , assume

$$\text{that } f^*(u_i u_j) = f^*(u_k u_j)$$

$$\Rightarrow i + n - 3 = k + n - 3$$

$$\Rightarrow i = k, \text{ a contradiction}$$

$$\Rightarrow \text{the edge labels are all distinct.}$$

*Case – 3:* For some  $i_1$  in  $1 \leq i \leq n - 3$  and  $i_2$  in

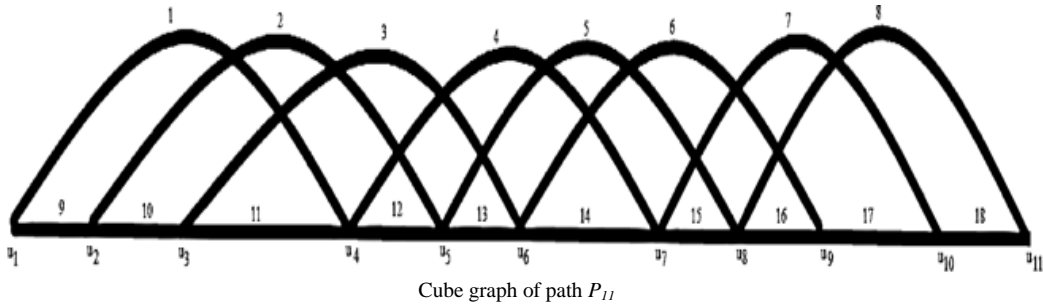
$$1 \leq i \leq n - 1, \text{ assume that } f^*(u_{i_1} u_j) = f^*(u_{i_2} u_j)$$

$$\Rightarrow i_1 = i_2 + n - 3,$$

$$\Rightarrow i_1 - i_2 = n - 3, \text{ a contradiction}$$

$$\Rightarrow \text{the edge labels are all distinct.}$$

From the claim discussed above, it has been proved that the edge labels of the graph are all distinct. The label of a vertex is defined as the sum of labels of the edges that are incident in it. Since the labels of the edges are all distinct, it follows that the vertex labels are also distinct. Hence the cube of path graph  $G(P_n^3)$ ,  $n = 11 + 4k$  and  $n \neq 9 + 4k$ ,  $k \in \mathbb{Z}^*$  admits vertex antimagic edge labeling. As an example, the graph of  $G(P_{11}^3)$  given by;



In the above cube of path graph, the edge labels are defined based on the function  $f^*: E(G) \rightarrow N$  as

$$f^*(u_i u_j) = i, j = i + 3, 1 \leq i \leq 8$$

$$f^*(u_i u_j) = i + 8, j = i + 1, 1 \leq i \leq 10$$

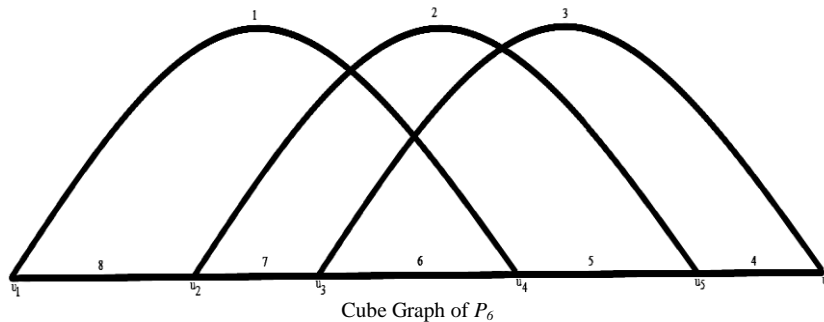
The above labeling pattern implies that the edge labels are distinct. The label of a vertex is the sum of labels of the edges that are incident to it. As the edge labels are distinct, the vertex labels are distinct too. Hence the graph  $G(P_{11}^3)$  admits vertex antimagic edge labeling.

A. Remarks:

The graph cube of a path  $P_n, n = 9 + 4k$  does not admit vertex antimagic edge labeling as the label of the vertices are not unique.

From the definition of the cube graphs, the cube graph of path  $P_4$  is a regular graph with each vertex having degree 2. The vertex antimagic edge labeling of regular graphs have already been proved.

The cube graph of the path  $P_6$  admits vertex antimagic edge labeling and the graph has been given as a particular case



In the above cube of path graph, the edge labels are defined based on the function  $f^*: E(G) \rightarrow N$  as

$$f^*(u_1 u_4) = 1, f^*(u_2 u_5) = 2, f^*(u_3 u_6) = 3,$$

$$f^*(u_6 u_5) = 4, f^*(u_5 u_4) = 5, f^*(u_4 u_3) = 6,$$

$$f^*(u_3 u_2) = 7 \text{ and } f^*(u_2 u_1) = 8$$

The above pattern of labeling shows that the edge labels are distinct and so too are the vertex labels. Hence the graph  $G(P_6^3)$  admits vertex antimagic edge labeling.

We tabulate the findings as follows;

Table 1: Inference Derived Out Of Labeling Pattern

S.No	Number of Vertices	Function Defined	Inference
1	$n \geq 5, n \neq 6$ $n \neq 11 + 4k$ $n \neq 9 + 4k, k \in Z^*$	$f^*(u_i u_j) = i, j = i + 1, 1 \leq i \leq n - 1$ $f^*(u_i u_j) = i + (n - 1), j = i + 3, 1 \leq i \leq n - 3$	Graph admits vertex antimagic edge labeling
2	$n = 6$	Labeling pattern is unique	Graph admits vertex antimagic edge labeling
3	$n = 11 + 4k, k \in Z^*$	$f^*(u_i u_j) = i, j = i + 3, 1 \leq i \leq n - 3$ $f^*(u_i u_j) = i + (n - 3), j = i + 1, 1 \leq i \leq n - 1$	Graph admits vertex antimagic edge labeling
4	$n = 9 + 4k, k \in Z^*$	The graph does not admit vertex antimagic edge labeling as the label of the vertices are not unique	

VI. CONCLUSION

In this paper, we have proved that the graph cube of a path denoted by  $G(P_n^3), n \geq 5, n = 6$  and  $n = 11 + 4k$  admits vertex antimagic edge labeling. The edges of the graph are labelled by two different labeling patterns. Two different theorems are stated to prove the admittance of antimagic labeling. The admittance of vertex antimagic edge labeling for the cube graph of path  $G(P_6^3)$  is shown separately as a particular case since the labeling of the edges of the graph is done in a specific manner. Also, the graph cube of a path on  $n = 9 + 4k, k \in Z^*$  does not admit the vertex antimagic edge

labeling. Similarly, the admittance of vertex antimagic edge labeling of the graph  $G(P_n^m), m \geq 4$  and  $m \in Z^*$  can also be proved.

REFERENCES:

1. R. Bodendiek and G. Walther – On arithmetic antimagic edge labelling of graphs, Mitt. Math.Ges. Hamburg 17 (1998), 85 – 89
2. Joseph. A. Gallian– A Dynamic Survey of Graph Labeling, The Electronic Journal of Combinatorics (2018), #DS6
3. Genghua, H. Fan and H.A. Kierstead – The Square of Paths and Cycles, Journal of Combinatorial Theory, Series B, Volume 63, Issue 1, January 1995, pg 55-61



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4. N. Hartsfield and G. Ringel – Pearls in Graph Theory, Academic Press, Boston – San Diego – New York – London, 1990
5. Martin Baca and Mirka Miller – Super Edge Antimagic Graphs – A Wealth of Problems and Some Solutions – Brown Walker Press, June 2008 , ISBN – 10: 1-59942-466-5 (e-book) ISBN – 13: 978-1-59942-466-8 (e-book)
6. J.A. Macdougall, M. Miller, Slamin and W.D. Wallis – Vertex Magic Total Labelling of Graphs – Utilitas Math. 61 (2002), 3 – 21
7. J.A. Macdougall, M. Miller and W.D. Wallis – Vertex Magic Total Labelling of Wheels and Related Graphs– Utilitas Math. 62 (2002), 175 – 18
8. J. Sedlacek – On Magic Graphs– Math. Slovaca 26 (1976), 329 – 335.
9. R. Sreenivasan, M.S. Paulraj – Vertex Antimagic Edge Labeling of Square of a Path Graph – Journal of Interdisciplinary Cycle Research, ISSN No: 0022-1945, Volume XI, Issue XI, November 2019, pg 761 – 763

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