New Fixed Point Results In Generalized G_b-Metric Spaces

R. K Saini, Mukesh Kushwaha, Adesh Kumar Tripathi

Abstract. In this communication, we establish new fixed point theorem in G_b-metric spaces. Moreover we examine the results for existence as well as uniqueness, which are related to the G-metric space. Our results generalize distinguished results and the mapping satisfying such contraction mention in the literature. In this sense, our results provide extension as well as improvement in the results of G_b-metric space. Also, we give some examples which verify our results.

Keywords: Set-valued contraction, b-metric spaces, fixed point theory, generalized G_b-metric spaces.

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I. INTRODUCTION

The fixed point theory is very significant and practical in mathematics. Due to its important and simplicity, several authors extended it many different results of fixed point theory. Amini [2] generalized its results in quasi contraction maps and also identifies various results of fixed point theory. TV. An [3] expressed the results of stone type theorem on b-metric space and allow some deep understanding of Fuzzy b-metric, set valued quasi contractions and Suzuki-type fixed point results, as we can see in [8,9]. Nashine and Kadelburg [5] found some results over contraction mapping which are loyal for all notions in metric spaces. Arshad [4] derived some different results of metric spaces which plays very important role in fixed point in b-metric spaces. Subsequently, many authors extend and generalized these theorems in different directions. In this article, we give a new generalize metric spaces introduce by Hussain et al.[6,7] and that covers a huge class of topological spaces including dislocated metric spaces with Fatou's property, G_b-spaces, b-metric spaces, generalize metric spaces.

In this work, we establish some results of [1] for generalize G_b-metric spaces, of course three variable x, y, z will be consider for solution. Also, the obtained results are supported by an application and examples for the existence and uniqueness solution for integral type problems.

II. PRELIMINARIES

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Definition.1 let X be a non-empty set and $s \geq 1$ be a given real number. A function $d^*: E\times E \to \mathbb{R}^+$ is called b-metric [1, 2, 3] if it satisfies the following properties for each $u,v,w \in E$

a. $d^*(u,v)=0 \Leftrightarrow u=v$;

b. $d^*(u,v)=d^*(v,u)$;

c. $d^*(u,w)=s\left[d^*(u,v)+d^*(u,w)\right]$

The pair (E,d^*) is called a b-metric space.

Example.1 Let $E=l_p(\mathbb{R})$ with $0<p<\frac{1}{s}$ where $l_p(\mathbb{R})=\left\{\{u_n\}\subset \mathbb{R}:\sum_{n=1}^{\infty}|u_n|^p<\infty\right\}$ define $d^*:E\times E\to\mathbb{R}^+$ as $d^*(u,v)=\left(\sum_{n=1}^{\infty}|u_n-v_n|^p\right)^{1/p}$ where $u=\{u_n\}, v=\{v_n\}$ then d^* is a b -metric space with coefficient $s=2^{1/p}$.

2. Let $E=L_p\left[0,\frac{1}{s}\right]$ be the set of all real function $u^*(t), t\in\left[0,\frac{1}{s}\right]$ such that $\int_0^1|u^*(t)|^p<\infty$ with $0<p<\frac{1}{s}$ define $d^*:E\times E\to\mathbb{R}^+$ as: $d^*(u,v)=\left(\int_0^1|u^*(t)-v^*(t)|^p\right)^{1/p}$

Then d^* is b-metric with coefficient $s=2^{1/p}$. If we take $s=1$ then the results similar goes to b -metric space with the concept of metric space for more details see [8, 9, 10]

Definition.2 Let (E,d^*) be a b -metric space. A sequence $\{u_n\}$ in E is called

(i) Cauchy iff $d^*(u_l,v_k)\to 0$ as $l,k\to\infty$;

(ii)Convergent iff there exists $u\in E$ such that $d^*(u_n,u)\to 0$ as $n\to\infty$ i.e. $\lim_{n\to\infty}u_n=u$;

(iii)The b-metric space (E,d) is complete [12] if every Cauchy sequence is convergent.

III. MAIN RESULTS

In this section we collect the previous information and to show the generalization of G_b-metric spaces also extended some results of fixed point results

Definition.3 Let (E,d) be a G-metric space. A sequence $\{u_n\}$ in E is said to be

(i) Cauchy iff $G(u_l,u_k,u_p)\to o$ as $l,k,p\to\infty$;

(ii)Convergent iff there exists $u\in E$ such that $G(u_l,u,u)\to o$ as $l\to\infty$ i.e. $\lim_{l\to\infty} u_l = u$;

(iii) If every Cauchy sequence is convergent then G-metric space (E,d) is complete.

Definition.4 Let E be a non-empty set and $G_\gamma : E\times E\times E\to\left[0,\frac{1}{s}\right)$ A mapping $G_\gamma : E\times E\times E\to\left[0,\frac{1}{s}\right)$ for $s=0$ all $u,v,w\in E$ satisfies the following condition.

$(G_\gamma 1)$: $G(u,v,w)=0$ iff $u=v=w$;

$(G_\gamma 2)$: $0<G(u,v,w)$ for all $u, v, w\in E$ with $u=v$;

$(G_\gamma 3)$: $G(u,u,u)\le G(u,v,w)$ for all $u, v, w\in E$ with $v\ne w$;

$(G_\gamma 4)$: $G(u,v,w)=G(u,w,v)=G(v,w,u)=G(v,w,u)=..$ (symmetry in all three variables);

$(G_\gamma 5)$: $G(u,v,w)\le G(u,a,a)+G(a,v,w)$ for all $u, v, w, a\in E$.

Then G is called a G-metric on E and (E, d) is called a G-metric space.

Remark.1 If we take $s=0$ then interval will convert into infinity.

Remark2 If $G(u,v,w)=s$ for s≥1 is G_b -metric space.

Example.2 Let $E=C([a,b],\mathbb{R}^+)$ be the set of all real valued functions define on [a, b]. E is complete extended G_b -metric space,

$$G_\gamma(u,v,w)=\sup_{t\in[a,b]}\left\{\begin{matrix}|u_1(t)-v_1(t)|+|u_2(t)-v_2(t)|+\\|v_1(t)-w_1(t)|+|v_2(t)-w_2(t)|\end{matrix}\right\}^2$$

where $\gamma(u,v,w)=|u(t)|+|v(t)|+|w(t)|+3$ and $\gamma : E\times E\times E\to\mathbb{R}^+$.

Theorem1 Let (E,d) be a complete extended G_b -metric space such that G_γ is a continuous function. Let $f:E\to E$ satisfying

$$G_\gamma(T_x,T_y,T_z)\le sG_\gamma(x,y,z),\ \forall x,y,z\in E \qquad (1)$$

Where $s\in[0,1]$ be such that for each $x_0\in E$, $\lim_{l,k,p\to\infty}\gamma(x_l,x_k,x_p)<\frac{1}{s}$, here $x_n=T^n x_0$, $n=1,2,3...$. Then T has a fixed point λ_1 since for each $y\in E$, $T^n y\to\lambda_1$.

Proof: Let $x_0\in E$ then by the definition of iteration scheme

$fx_0=x_1, x_2=f(x_1)=f(Tx_0)=f^2(x_0),...,x_n=f^n(x_0)$

According equation (1), we proceeds the approximation then we have

$$G_\gamma(x_l,x_{l+1},x_{l+2})\le sG_\gamma(x_0,x_1,x_2) \qquad (2)$$

$$G_\gamma(x_l,x_k,x_p)\le\begin{cases}\gamma(x_l,x_k,x_p)s^nG_\gamma(x_0,x_1,x_2)+\\\gamma(x_l,x_k,x_p)\gamma(x_{l+1},x_k,x_p)s^{n+1}G_\gamma(x_0,x_1,x_2)+....\\+\gamma(x_{k-2},x_k,x_p)\gamma(x_{k-1},x_k,x_p)s^{k-1}G_\gamma(x_0,x_1,x_2)\end{cases}$$

Therefore the series $\sum_{n=1}^{\infty}s^n\prod_{i=1}^{n}\theta(x_i,x_k)$ is convergent. Moreover,

$G_\gamma(x_l,x_k,x_p)\le G_\gamma(x_0,x_1,x_2)\{[a_l-a_k]+[a_k-x_l]\}$

Let us assume $\{x_n\}$ be any sequence. Since E is complete. Let $x_l\to R\in E$

$$G_\gamma(TR,R,R)\le\gamma(TR,R,R)\begin{bmatrix}G_\gamma(TR,x_l,x_k)+G_\gamma(x_l,R,x_l)\\+G_\gamma(x_l,x_k,R)\end{bmatrix}$$

$$\le\gamma(TR,R,R)\begin{bmatrix}G_\gamma(R,x_{l-1},x_{k-1})+G_\gamma(x_l,x_k,R)\\+G_\gamma(x_l,R,x_k)\end{bmatrix}$$

$G_\gamma(TR,R,R)\le 0$ as $l,k\to\infty$,then $G_\gamma(TR,R,R)=0$

Here sequence $\{x_l\}$ has a fixed point R as we can see the results of Kamran et al. [1] using equation (1)

Example.3. $G_\gamma(x,y,z):E\times E\times E\to\mathbb{R}^+$ and as: $G_\gamma(x,y,z)=(x-y-z)^2$, where $\gamma(x,y,z)=x+y+z+3$ then G_γ is called extended G_b -metric on E. Let us define $T:E\to E$ by $Tx=\frac{x+1}{2}$ we have

$$G_\gamma(Tx,Ty,Tz)=\left\{\left(\frac{x+1}{2}+\frac{y+1}{2}\right)-\left(\frac{y+1}{2}+\frac{z+1}{2}\right)\right\}^2\le sG_\gamma(x,y,z)$$

note that for each $x\in E$, $T^l x=\frac{x+1}{2^{l+1}}$. Thus we obtain:

$$\lim_{l,k,p\to\infty}=\gamma(T^lx,T^kx,T^p)=\lim_{l,k,p\to\infty}\left(\frac{x+1}{2^{l+1}}+\frac{y+1}{2^{k+1}}+\frac{z+1}{2^{p+1}}+3\right)<4$$

Hence T has a fixed point.

IV APPLICATION

Consider the function

$$\gamma(t)=\int_a^b G(t,\gamma(s))ds \qquad (3)$$

Where $k:E\times\mathbb{N}\to[0,\infty)$ is continuous function where $E=[a,b]$ now, define the function $\delta:E\to E$ express it by

$$\delta\gamma(t)=\int_a^b G(t,\gamma(s))ds \qquad (4)$$

Mention in equation (2) has a fixed point solution thus we must say T has a solution.

Theorem.2 Consider the integral [0, 1] where $\lambda\in[0,1]$ such that for every $s\in[0,1]$ and $u_1,u_2\in E$ we have

$G\left(s,u_1(s),u_2(s)\right) \le \frac{\lambda}{b-a}\left[u_1(s)-u_2(s)\right]$ then T has a fixed point in E.

Proof: Here for all $x \in E$, therefore

$$\left|\delta(x)(t)\right| \le \int_a^b \left|G\left(t,s,u_1,u_2\right)\right| ds \le \frac{\lambda}{b-a}\int_a^b \left|u_1(s)-u_2(s)\right| ds$$

$$\le \lambda \left\|u_1(s)-u_2(s)\right\|$$

It follows that for all $x, y, z \in E$,

$$G_\gamma\left(Tx,Ty,Tz\right) \le \lambda G\left(x,y,z\right). \qquad (5)$$

Hence T has a fixed point, where $\left\|u_1-u_2\right\| = u \in E$.

In the similar way we can also show the results of Voltera and Fredholm integral type solution.

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