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Inequalities Between First and Second Order Moments for Continuous Probability Distribution

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***Abstract:** In this research article we obtained some inequalities between moments of 1st and 2nd order for a continuous distribution over the interval [x, y], when infimum and supremum of the continuous probability distribution is taken into consideration. These inequalities have shown improvement and are better than those exist in literature. Inequalities also obtained for continuous random variables which vary in [x, y] interval, such that the probability density function (pdf) $\phi(t)$ become zero in [p, q] $\subset$ [x, y].The improvement in inequalities have been shown graphically. Here in this paper we deduced some existing inequalities by using the inequalities obtained in Theorem 2.1 and Theorem 2.2.*

***Key Words:** Moments, Variance, Random Variable (R.V.), Infimum and Supremum*

I. INTRODUCTION

If μ_1' be mean, σ^2 be variance and μ_2' the 2nd order moment, the for R.V. in the interval [x, y] , we have following:

$$\mu_1' = \int_x^y t\phi(t)dt \qquad (1.1)$$

$$\sigma^2 = \int_x^y (t-\mu_1')^2\phi(t)dt \quad \text{and} \qquad (1.2)$$

$$\mu_2' = \int_x^y t^2\phi(t)dt . \qquad (1.3)$$

Relationship between mean (μ_1'), variance (σ^2) and second order moment about origin (μ_2') is:

$$\mu_2' = \sigma^2 + \mu_1'^2 . \qquad (1.4)$$

Bounds of the variance ($\sigma^2 = \mu_2' - \mu_1'^2$) of a R.V., its extensions and applications have been found in literature.

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These inequalities have been discussed extensively over the period of time by B.C. Rennie, "On a class of Inequalities"(1963), D.S. Mitrinovic, "Analytic Inequalities" (1970), J.N. Kapur et al (1995), M.B. Benerjee et. al (1995), J.N. Kapur (1996), N.S. Barnett et al (1999), R. Bhatia and C. Davis in(2000), R Sharma et. al (2010). J.N. Kapur in (1996) obtained the following inequality when R.V. fall in the interval [x, y]:

$$0 \le \sigma^2 \le (y-\mu_1')(\mu_1'-x) \qquad (1.5)$$

(1.5) can be written as

$$0 \le \sigma^2 + \left(\mu_1' - \frac{x+y}{2}\right)^2 \le \left(\frac{y-x}{2}\right)^2 . \qquad (1.6)$$

J.N. Kapur in (1995) has shown that the 2nd order moment μ_2' about origin lies between following maximum and minimum values:

$$\mu_1'^2 \le \mu_2' \le (x+y)\mu_1' - xy . \qquad (1.7)$$

Inequality (1.5) and (1.7) are equivalent.

The plot of inequality (1.7) shows that point (μ_1', μ_2') lies in μ_1' μ_2'- plane region between quadratic and linear curve of moments as shown in figure (1.1).

Such type of basic inequalities their extensions, refinements was studied by different mathematicians. Alternative proofs for these inequalities also given by several authors. In particular Sharma et. al (2012) had given proofs for the inequalities involving moments of discrete uniform distributions.

$$\mu_2' \ge \frac{x^2+y^2}{N} + \frac{N}{N-2}\left[\mu_1' - \frac{x+y}{N}\right]^2 .$$

This inequality shows refinement in inequality (1.7) given by J N Kapur in (1996). Here in this paper first two moments μ_1' and μ_2' of a R.V. which is continuous and which lie in interval [x, y] are considered. We derive some inequalities which involve 1st and 2nd order moments about origin for a continuous probability distribution in the presence of minimum and maximum values of the pdf when R.V. lies in the interval [x, y]. A bound on variance of a R.V. which lies in the interval [x, y] was discussed by Kapur et. al, (1996) and is given by inequalities (1.5),(1.6) and (1.8).

Here we have obtained some refinements in (1.7) subject to the condition that pdf is taken into consideration (both minimum and maximum values) and also discussed related refinements and extensions. The main results (Theorem 2.1 and Theorem 2.2) gives bound for 2nd order moment and 1st order moment subject to above condition that also shows refinement in (1.7).

Sharma et. al. (2015) proved following inequalities (1.8) and (1.9) below for the case when, mean is μ_1' and standard deviation is σ for R.V. "t" ϕ(t) is pdf defined in [x, y], m be the infimum and M is supremum of the function ϕ(t) in the interval [x, y], then

$$\frac{m(y-x)^3}{12} \le \sigma^2 + \left(\mu_1' - \frac{x+y}{2}\right)^2 \le \left(\frac{y-x}{2}\right)^2 - \frac{m(y-x)^3}{6} \quad (1.8)$$

and

$$\left(\frac{y-x}{2}\right)^2 - \frac{M(y-x)^3}{6} \le \sigma^2 + \left(\mu_1' - \frac{x+y}{2}\right)^2 \le M\frac{(y-x)^3}{12} \quad (1.9)$$

In Theorem 2.3 it has been shown that above bounds can be deduced by using Theorem 2.1 and Theorem 2.2 below.

We also discuss the results i.e. in theorem 2.1 and 2.2 geometrically. Theorem 2.4 shows how the region in figure 1.1 for inequality (1.7) reduces to new region for new inequality and shown by figure 1.2.

II. MAIN RESULTS

THEOREM 2.1

Let μ_1' and μ_2' be the moments, ϕ(t) be pdf and "t" a continuous R.V. whose pdf. is defined in the interval [x, y]. If m be the greatest lower bound for pdf ϕ(t) in [x, y] then we must have,

$$(x+y)\mu_1' - \frac{(x+y)^2}{4} + \frac{m(y-x)^3}{12} \le \mu_2' \le (x+y)\mu_1' - xy - \frac{m(y-x)^3}{6} \quad (1.10)$$

PROOF

Consider

$$\int_x^y [t^2 - (x+y)t + xy]\phi(t)dt = \int_x^y (t-y)(t-x)\phi(t)dt.$$

Since t lies in the interval [x, y] therefore,

(t – x) (t – y) ≤ 0

$$\Rightarrow \int_x^y [t^2 - (x+y)t + xy]\phi(t)dt \le m\int_x^y (t-y)(t-x)dt$$

$$\Rightarrow \mu_2' \le (t+y)\mu_1' - ty - \frac{m(y-x)^3}{6}. \quad (1.11)$$

Now $t^2 - (t+y)t + ty = \left(t - \frac{x+y}{2}\right)^2 - \left(\frac{y-x}{2}\right)^2$

$$\Rightarrow \int_x^y \left(t^2 - (x+y)t + \frac{(y+x)^2}{4}\right)\phi(t)dt = \int_x^y \left(t - \frac{x+y}{2}\right)^2 \phi(t)dt. \quad (1.12)$$

Using properties of pdf ϕ(t) over the limits x to y in (1.12) we have

$$\Rightarrow \mu_2' \ge (t+y)\mu_1' - \left(\frac{x+y}{2}\right)^2 + m\frac{(y-x)^3}{12}. \quad (1.13)$$

Inequality (1.10) now follows from inequality (1.11) and (1.13).

By using algebraic operations and techniques of derivatives for maxima and minima we can easily show that

$$\mu_1'^2 \le (x+y)\mu_1' - \frac{(x+y)^2}{4} + \frac{m(y-x)^3}{12} \le \mu_2' \le (x+y)\mu_1' - xy - \frac{m(y-x)^3}{6} \le (x+y)\mu_1' - xy.$$

This shows refinement in (1.7).

THEOREM 2.2

Let μ_1' and μ_2' be moments of a continuous R.V. "t" whose pdf ϕ(t) is defined in [x, y]. If M be the least upper bound of the function ϕ(t) in [x, y] then we must have,

$$(x+y)\mu_1' - xy - \frac{M(y-x)^3}{6} \le \mu_2' \le (x+y)\mu_1' - \left(\frac{x+y}{2}\right)^2 + \frac{M(y-x)^3}{12} \quad (1.14)$$

ROOF

Consider

$$\int_x^y [t^2 - (x+y)t + xy]\phi(t)dt = \int_x^y (t-y)(t-x)\phi(t)dt.$$

Since t ε [x, y] hence we have:

(t – x) (t – y) ≤ 0

$$\Rightarrow \int_x^y [t^2 - (x+y)t + xy]\phi(t)dt \ge M\int_x^y (t-y)(t-x)dt$$

$$\Rightarrow \mu_2' \ge (t+y)\mu_1' - xy - \frac{M(y-x)^3}{6}. \quad (1.15)$$

Now

$$t^2 - (x+y)t + xy = \left(t - \frac{x+y}{2}\right)^2 - \left(\frac{y-x}{2}\right)^2$$

$$\Rightarrow \int_x^y \left(t^2 - (x+y)t + \left(\frac{y+x}{2}\right)^2\right)\phi(t)dt = \int_x^y \left(t - \frac{x+y}{2}\right)^2 \phi(t)dt \quad (1.16)$$

On simplifying (1.16) we have

$$\mu_2' \le (x+y)\mu_1' - \left(\frac{y+x}{2}\right)^2 + \frac{M(y-x)^3}{12} \qquad (1.17)$$

Inequality (1.14) now follows from inequality (1.15) and (1.17).
Similarly (1.14) is refinement in (1.7) and is given as

$$\mu_1'^2 \le (x+y)\mu_1' - xy - \frac{M(y-x)^3}{6} \le \mu_2' \le$$

$$(x+y)\mu_1' - \left(\frac{x+y}{2}\right)^2 + \frac{M(y-x)^3}{12} \le (x+y)\mu_1' - xy$$

THEOREM 2.3
If t be a R.V. in [x, y] and $\phi(t)$ be pdf then inequalities (1.8) and (1.9) can be deduced from (1.10) and (1.14) respectively.

PROOF
We know that μ_1', μ_2' and σ are related by the following equation:

$$\mu_2' = \sigma^2 + \mu_1' 2 \quad . \qquad (1.18)$$

Substituting value of μ_2' from (1.18) in inequality (1.10), we get inequality (1.8). Similarly substituting the value of μ_2' from (1.18) in inequality (1.14), we get inequality (1.9).

THEOREM 2.4
If t be a R.V. which lies in [x,y], with pdf $\phi(t)$, such that $\phi(t)$ vanishes in $[p, q] \subset [x, y]$, we must have,

$$(p+q)\ \mu_1' - pq \le \mu_2' \le (x+y)\ \mu_1' - xy. \qquad (1.19)$$

From inequality (1.19) it is obvious that point (μ_1', μ_2') in $\mu_1'\ \mu_2'$ - plane lies on or below the curve

$\mu_2' = (x + y)\ \mu_1' - xy$ and on or above the curve

$\mu_2' = (p+q)\ \mu_1' - pq$.

This is shown in figure (1.2).

PROOF
Since $(t-x)(t-y) \le 0$ for t in [x, y] then,

$$t^2 \le (x+y)\, t - xy \qquad (1.20)$$

If 't' lies outside i.e. $t \notin [p, q]$ then,

$(t-p)(t-q) \ge 0$

$$t^2 \ge (p+q)\, t - pq \qquad (1.21)$$

Multiply (1.20) with $\phi(t)$ the pdf and integrating with R.V. t between limits t = x and t = y we get maxima for inequality (1.19). Similarly, on multiply (1.21) by pdf $\phi(t)$ and on integrating minima for inequality (1.19).

When $\phi(t)$ vanishes in $[p, q] \subset [x, y]$ the point (μ_1', μ_2') lies on or below the line joining points E and F. Further the point (μ_1', μ_2') lies on or above the line joining points C and D. Figure (1.2) shows that inequality (1.19) is refinement of inequality (1.7) shown by Figure (1.1).

III. CONCLUSIONS

Theorem 2.1 and Theorem 2.2 shows that the results obtained are better than inequality (1.7).Theorem 2.3 proved that inequality (1.8) and (1.9) obtained by Sharma et..al. in (2015) can be deduced from inequalities obtained in theorem2.1 and 2.2 by using the relation $\mu_2' = \sigma^2 + \mu_1'^2$. Theorem 2.4 proved that inequality (1.19) is better than (1.7) when pdf ϕ (t) assume zero value in the interval [p,q] contained in [x, y]. Betterment in inequality (1.19) explained geometrically and it shows that shaded region given by inequality (1.19) in figure 1.2 is more compact than in figure 1.1 for inequality (1.7).

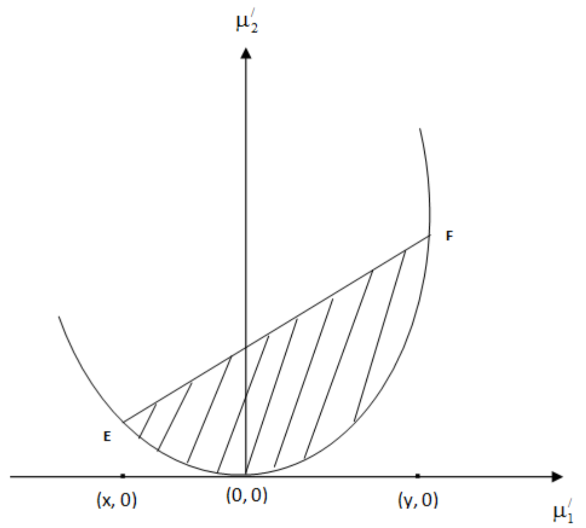

Figure: 1.1
$E \leftrightarrow (x, x^2)$ and $F \leftrightarrow (y, y^2)$.

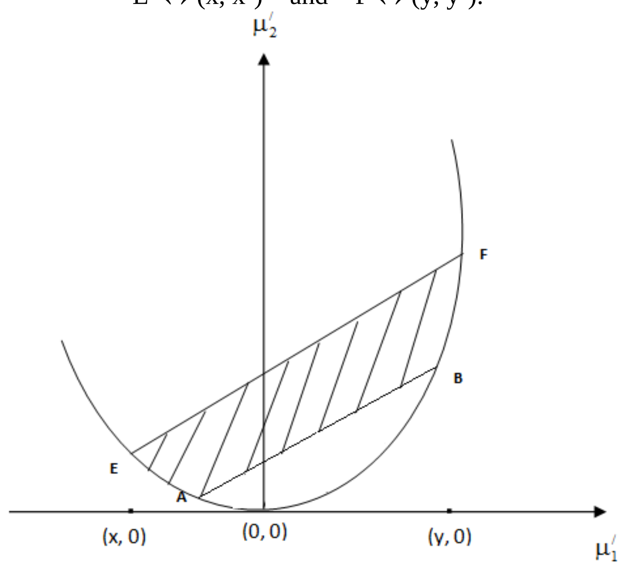

Figure: 1.2
$A \leftrightarrow (p, p^2)$ $B \leftrightarrow (q, q^2)$

$E \leftrightarrow (x, x^2)$ and $F \leftrightarrow (y, y^2)$

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