Robust Fault Estimation and Fault-Tolerant Control Based on Sliding Mode Observer for Takagi–Sugeno Fuzzy Systems Subject to Actuator and Sensor Faults

Slim Dhahri, Essia Ben Alaïa

Abstract: ***In this paper, the problems of fault estimation and fault-tolerant control for Takagi-Sugeno fuzzy system affected by simultaneous actuator faults, sensor faults and external disturbances are investigated. Firstly, an adaptive fuzzy sliding-mode observer is designed to simultaneously estimate system states and both actuator and sensor faults. Then, based on the online estimation information, a static output feedback fault-tolerant controller is designed to compensate for the effect of faults and to stabilize the closed-loop system. Moreover, sufficient conditions for the existence of the proposed observer and controller with an H_∞ performance are derived based on Lyapunov stability theory and expressed in terms of linear matrix inequalities. Finally, a nonlinear inverted pendulum with cart system application is given illustrate the validity of the proposed method.***

Keywords: Fault estimation, sliding mode observer, static output feedback fault-tolerant controller, Takagi-Sugeno fuzzy system.

I. INTRODUCTION

In modern control systems, the increasing demand of higher performance, safety, reliability, maintainability, and survivability represent a major concern. Thus, it is important to encourage the development of research on fault-tolerant control (FTC) in order to guarantee these objectives. Many outstanding results have been achieved during last two decades, such as those reported in [1]–[4] and the references therein. The main results on the FTC can be classified into two main strategies: the first one, the so-called passive FTC, is focused on to conceive a robust controller against disturbances and uncertainties. A key limitation is that the closed-loop system stability can't be achieved even faults occurrence. Nevertheless, active FTC is important and it takes a primordial place in modern control application. In practical applications, most of the control systems usually have nonlinear behaviors.

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Takagi-Sugeno (T–S) fuzzy systems provide a powerful technique to approximate a large class of nonlinear dynamic systems. T-S fuzzy systems are nonlinear models represented by a set of local linear models. By fuzzy blending of the linear system representations the overall fuzzy model of the system is achieved, which greatly facilitates observer/controller synthesis for complex nonlinear systems. Thus, excellent papers about robust FE and FTC problem of T-S system subject to actuator or sensor faults are developed in [5]–[8].

In practical engineering, actuator and sensor faults may occur simultaneously and disturbances may exist [9]–[14]. So far, a new robust fuzzy scheduler FTC has been developed in [14] for nonlinear systems affected by simultaneous sensor faults, actuator faults and parameter uncertainties. Some sufficient conditions for robust stabilization have been derived and formulated in the linear matrix inequalities (LMIs) format. A proportional integral (PI) observer has been proposed in [10] to simultaneous estimation of states, time varying sensor and actuator faults for T-S fuzzy model. Reference [9] shows FE and observer-based FTC scheme for T–S fuzzy systems using local nonlinear models and having sensor and actuator faults, simultaneously. A new descriptor fuzzy sliding mode observer (SMO) has been designed in [11] to estimate the system state and sensor and actuator faults simultaneously, and an observer-based FTC scheme has been developed to stabilize a closed-loop system. A new structure of T-S SMO with two discontinuous terms has been developed in [12], [13] to solve the problem of simultaneous actuator and sensor faults reconstruction for a class of uncertain T-S nonlinear system with unmeasurable premise variables.

There are, so far as the authors know, no works dealing with adaptive SMO-based static output-feedback fault tolerant controller (SOFFTC) for T-S systems considering simultaneously actuator, sensor faults and external disturbances. Motivated by the above observations, an adaptive SMO using H_∞ optimization technique is developed to estimate both states and sensor/actuator faults. Then a SOFFTC is designed to guarantee the stability of the closed-loop faulty system. Sufficient conditions for the existence of SMO and SOFFTC were given in terms of LMIs. The observer and controller are designed separately, which avoids their coupling and reduces the computation complexity.

The rest of this paper is organized as follows. Problem description is represented in Section 2. Design of augmented system is introduced in Section 3. Robust adaptive fuzzy SMO design and stability analysis of the error dynamics are given in Section 4. Fault estimation approach is studied in Section 5. Robust static output feedback fault tolerant control strategy is addressed in Section 6. Simulation results on nonlinear inverted pendulum with cart system are developed in Section 7 validates the efficiency of the proposed algorithm. Some conclusion are given in Section 8.

II. PROBLEM DESCRIPTION

Consider the T–S fuzzy model with additive actuator and sensor faults. The *i*th rule of the T-S fuzzy model is of the following form:

Plant Rule *i*: IF ζ_1 is $\Delta_{1,i}$ and ... ζ_g is $\Delta_{g,i}$, THEN

$$\dot{x} = A_i x + B_i u + M_i f_a + E_i d \ , \tag{1}$$

$$y = Cx + Nf_s \ , \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$ denote, respectively, the state vector, the input vector and the measurement output vector. $f_a \in \mathbb{R}^q$ and $f_s \in \mathbb{R}^h$ represent additive actuator fault vector and sensor fault vector, respectively. $d \in \mathbb{R}^l$ represents the exogenous disturbance vector, which is assumed to belong to $\mathcal{L}_2[0,\infty)$. A_i, B_i, M_i, E_i, C and N are known constant matrices with appropriate dimensions. It is supposed that matrix C is of full row rank, N is of full column rank, the pairs (A_i, B_i) are controllable, the pairs (A_i, C) are observable, and $n > p \geq q + h$. Z_j *(j=1,...,g)* are the premise variables which are assumed to be measurable, and $\Delta_{g,i}$ *(j=1,...,g;i=1,...,k)* are fuzzy sets that are characterized by membership function. *g* and *k* are the number of premise variables and IF-THEN rules, respectively.

The fuzzy systems can be written as follows:

$$\dot{x} = \sum_{i=1}^{k} \mu_i \left(A_i x + B_i u + M_i f_a + E_i d \right) \ , \tag{3}$$

$$y = Cx + Nf_s \ , \tag{4}$$

where $\zeta = [\zeta_1, ..., \zeta_g]$. For each i = 1, 2,..., k, μ_i is the abbreviation of $\mu_i(\zeta)$, where $\mu_i(\zeta)$ is the weighting function defined as:

$$\mu_i(\zeta) = \frac{w_i(\zeta)}{\sum_{i=1}^{k} w_i(\zeta)}, \quad w_i(\zeta) = \prod_{j=1}^{g} \Delta_{ij}(\zeta_j) \ , \tag{5}$$

and $\Delta_{ij}(\zeta_j(t))$ is the grade of the membership function of ζ_j. It is obvious that

$$w_i(\zeta) \geq 0, i = 1, ..., k, \text{ and } \sum_{i=1}^{k} w_i(\zeta) > 0 \ . \tag{6}$$

For any ζ. Hence the normalized fuzzy membership function satisfies

$$\mu_i(\zeta) \geq 0, i = 1, ..., k, \text{ and } \sum_{i=1}^{k} \mu_i(\zeta) = 1 \ . \tag{7}$$

Assumption 1. The actuator fault fa and sensor fault fs satisfy

$$\| f_a \| \leq \rho_a, \| f_s \| \leq \rho_s, f = \begin{bmatrix} f_a^T & f_s^T \end{bmatrix}^T \leq \rho_{as} \ , \tag{8}$$

where ρ_a, ρ_s and ρ_{as} are three known positive constants.

Assumption 2. The actuator fault distribution matrices Mi in (3) satisfies:

$$rank(CM_i) = rank(M_i) = q \ . \tag{9}$$

Assumption 3.

$$rank \begin{bmatrix} sI_n - A_i & M_i \\ C & 0 \end{bmatrix} = n + q \ . \tag{10}$$

for all complex number s with $Re(s) \geq 0$.

AUGMENTED SYSTEM

In order to partition the output vector into non-faulty and potentially faulty components, we assume that there exists an orthogonal matrix $T_r \in \mathbb{R}^{p \times p}$ such that

$$T_r C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, T_r N = \begin{bmatrix} 0 \\ N_1 \end{bmatrix} , \tag{11}$$

where $N_1 \in \mathbb{R}^{h \times h}$ is a nonsingular matrix. Scaling the measured output y(t) by Tr, yields

$$\begin{cases} y_1 = C_1 x \\ y_2 = C_2 x + N_1 f_s \end{cases} , \tag{12}$$

where $y_1 \in \mathbb{R}^{p-h}$ and $y_2 \in \mathbb{R}^h$.

If we introduce y_2 satisfying the following differential equation

$$\begin{aligned} \dot{x}_f &= -A_f x_f + A_f y_2, \\ &= -A_f x_f + A_f C_2 x + A_f N_1 f_s, \end{aligned} \tag{13}$$

where $-A_f \in \mathbb{R}^{h \times h}$ is stable. An augmented system can be obtained:

$$\dot{\bar{x}} = \sum_{i=1}^{k} \mu_i \left(\bar{A}_i \bar{x} + \bar{B}_i u + \bar{M}_i f + \bar{E}_i d \right) , \tag{14}$$

$$\bar{y} = \bar{C}\bar{x} \ , \tag{15}$$

where

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$$\bar{x}=\begin{bmatrix} x \\ x_f \end{bmatrix}, f=\begin{bmatrix} f_a \\ f_s \end{bmatrix}, \bar{y}=\begin{bmatrix} y_1 \\ x_f \end{bmatrix}, \quad (16)$$

with

$$\bar{A}_i=\begin{bmatrix} A_i & 0 \\ A_f C_2 & -A_f \end{bmatrix}, \bar{B}_i=\begin{bmatrix} B_i \\ 0 \end{bmatrix}, \bar{M}_i=\begin{bmatrix} M_i & 0 \\ 0 & A_f N_1 \end{bmatrix},$$
$$\bar{E}_i=\begin{bmatrix} E_i \\ 0 \end{bmatrix}, \bar{C}=\begin{bmatrix} C_1 & 0 \\ 0 & I_h \end{bmatrix}. \quad (17)$$

Lemma1. The condition

$$rank\left(\bar{C}\bar{M}_i\right)=rank\left(\bar{M}_i\right)=q+h, \quad (18)$$

holds if and only if (9) holds.

Proof.

From (17), we have

$$\bar{C}\bar{M}_i=\begin{bmatrix} C_1 & 0 \\ 0 & I_h \end{bmatrix}\begin{bmatrix} M_i & 0 \\ 0 & A_f N_1 \end{bmatrix}=\begin{bmatrix} C_1 M_i & 0 \\ 0 & A_f N_1 \end{bmatrix}. \quad (19)$$

Pre-multiply $\bar{C}\bar{M}_i$ in (19) with

$$\begin{bmatrix} I_{p-h} & 0 \\ 0 & A_f^{-1} \end{bmatrix}. \quad (20)$$

$rank\left(\bar{C}\bar{M}_i\right)=q+h$ will be satisfied if and only if

$$rank\begin{bmatrix} C_1 M_i & 0 \\ 0 & N_1 \end{bmatrix}=rank(T_r C\begin{bmatrix} M_i & 0 \\ 0 & N_1 \end{bmatrix})=q+h. \quad (21)$$

Since N1 is full rank and Tr is an orthogonal matrix, (21) will be satisfied if and only

$$rank\left(CM_i\right)=q. \quad (22)$$

So the condition (18) is satisfied for augmented system (14)-(15).

Lemma 2. The condition

$$rank\begin{bmatrix} sI_n-\bar{A}_i & \bar{M}_i \\ \bar{C} & 0 \end{bmatrix}=n+q+2h, \quad (23)$$

holds for all complex number s with $Re(s)\geq 0$ if and only if (10) holds.

Proof.

The invariant zeros of $(\bar{A}_i,\bar{M}_i,\bar{C})$ are given by the values of s when R(s) loses normal rank

$$R(s)=\begin{bmatrix} sI_n-\bar{A}_i & \bar{M}_i \\ \bar{C} & 0 \end{bmatrix}. \quad (24)$$

R(s) will lose rank if and only if

$$R(s)=\begin{bmatrix} sI_n-\bar{A}_i & \bar{M}_i \\ \bar{C} & 0 \end{bmatrix}, \quad (25)$$

loses rank. Pre-multiplying the matrix pencil in (25) by

$$\begin{bmatrix} I_n & 0 \\ 0 & T_r^{-1} \end{bmatrix}\begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_{p-h} \\ 0 & -A_f^{-1} & 0 \end{bmatrix}, \quad (26)$$

which satisfy

$$rank\begin{bmatrix} sI_n-A_i & M_i \\ C & 0 \end{bmatrix}\neq n+q, \quad (27)$$

ADAPTIVE SLIDING MODE OBSERVER DESIGN

The proposed adaptive fuzzy SMO has the structure

$$\dot{\hat{x}}=\sum_{i=1}^{k}\mu_i\left(\bar{A}_i\hat{x}+B_i u+\bar{G}_{l,i}e_y+\bar{G}_{n,i}\upsilon\right), \quad (28)$$

$$\hat{y}=\bar{C}\hat{x}, \quad (29)$$

where $\hat{x}$ is the estimate of $\bar{x}$, $\hat{y}$ is estimate of $\bar{y}$ and $\bar{e}_y:=\bar{y}-\hat{y}$. $\bar{G}_{l,i}$ and $\bar{G}_{n,i}$ are design matrices to be determined and υ represents a discontinuous switched term.

A canonical form

Under condition (18), there exists a transformation $\left(x_1^T,x_2^T\right)^T=\sum_{i=1}^{k}\mu_i T_i\bar{x}$ such that:

$$\dot{x}_1=\sum_{i=1}^{k}\mu_i\left(\bar{A}_{11,i}x_1+\bar{A}_{12,i}x_2+\bar{B}_{1,i}u+\bar{E}_{1,i}d\right), \quad (30)$$

$$\dot{x}_2=\sum_{i=1}^{k}\mu_i\left(\bar{A}_{21,i}x_1+\bar{A}_{22,i}x_2+\bar{B}_{2,i}u+\bar{M}_{2,i}f+\bar{E}_{2,i}d\right), \quad (31)$$

$$\bar{y}=\bar{C}_2 x_2, \quad (32)$$

where $x_1\in\mathbb{R}^{n+h-p}$ and $x_2\in\mathbb{R}^{p}$ are new state vectors, and

$$T_i\bar{A}_i T_i^{-1}=\begin{bmatrix} \bar{A}_{11,i} & \bar{A}_{12,i} \\ \bar{A}_{21,i} & \bar{A}_{22,i} \end{bmatrix}, T_i\bar{B}_i=\begin{bmatrix} \bar{B}_{1,i} \\ \bar{B}_{2,i} \end{bmatrix},$$
$$T_i\bar{E}_i=\begin{bmatrix} \bar{E}_{1,i} \\ \bar{E}_{2,i} \end{bmatrix}, \bar{C}T_i^{-1}=\begin{bmatrix} 0 & \bar{C}_2 \end{bmatrix}, \quad (33)$$

where $\bar{A}_{11,i}\in\mathbb{R}^{(n+h-p)\times(n+h-p)}$, $\bar{E}_{1,i}\in\mathbb{R}^{(n+h-p)\times l}$ and $\bar{C}_2\in\mathbb{R}^{p\times p}$ is nonsingular. The fault distribution matrix

$$T_i\bar M_i = \begin{bmatrix}0_{(n+h-p)\times(q+h)}\\ \bar M_{2,i}\end{bmatrix} = \begin{bmatrix}0_{(n+h-p)\times(q+h)}\\ 0\\ \bar M_{22,i}\end{bmatrix}, \quad (34)$$

where $\bar M_{2,i}\in\mathbb{R}^{p\times(q+h)}$ and $\bar M_{22,i}\in\mathbb{R}^{(q+h)\times(q+h)}$ is non singular.

Consider system (30)-(32). Introduce a further coordinate transformation $z(t)=\sum_{i=1}^{k}\mu_i T_{L,i}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}$ where

$$T_{L,i}=\begin{bmatrix}I_{n+h-p} & \bar L_i\\ 0_{p\times(n+h-p)} & \bar C_2\end{bmatrix}, \quad (35)$$

where $\bar L_i\in\mathbb{R}^{(n+h-p)\times p}$ has the structure

$$\bar L_i=\begin{bmatrix}\bar L_{1,i} & 0\end{bmatrix}, \quad (36)$$

with $\bar L_{1,i}\in\mathbb{R}^{(n+h-p)\times(p-(q+h))}$. Then, system (30)-(32) becomes following form

$$z_1=\sum_{i=1}^{k}\mu_i\left(\tilde A_{11,i}z_1+\tilde A_{12,i}z_2+\tilde B_{1,i}u+\tilde E_{1,i}d\right), \quad (37)$$

$$z_2=\sum_{i=1}^{k}\mu_i\left(\tilde A_{21,i}z_1+\tilde A_{22,i}z_2+\tilde M_{2,i}f+\tilde E_{2,i}d\right), \quad (38)$$

$$\bar y = z_2, \quad (39)$$

where $z:=col(z_1,z_2)$ with $z_1\in\mathbb{R}^{n+h-p}$. The matrix $\bar A_i$, in the new coordinate system, is given as

$$T_i\bar A_i T_i^{-1}=\begin{bmatrix}\tilde A_{11,i} & \tilde A_{12,i}\\ \tilde A_{21,i} & \tilde A_{22,i}\end{bmatrix}=\begin{bmatrix}\bar A_{11,i}+\bar L_i\bar A_{21,i} & (\bar A_{12,i}+\bar L_i\bar A_{22,i})\bar C_2^{-1}-(\bar A_{11,i}+\bar L_i\bar A_{21,i})\bar C_2^{-1}-\bar A_{11,i}\bar L_i\bar C_2^{-1}\\ \bar C_2\bar A_{21,i} & \bar C_2\left(\bar A_{22,i}-\bar A_{21,i}\bar L_i\right)\bar C_2^{-1}\end{bmatrix}, \quad (40)$$

By definition, $\tilde M_{2,i}=\bar C_2\bar M_{2,i}$, $\tilde E_{2,i}=\bar C_2\bar E_{2,i}$ and

$$\tilde C=\bar C T_{L,i}^{-1}=\begin{bmatrix}0 & I_p\end{bmatrix}, \quad (41)$$

Because of the special structure of $\bar L_i$ from (36), further partition $\bar A_{21,i}$ and $\bar E_{2,i}$ from (33) as

$$\bar A_{21,i}=\begin{bmatrix}\bar A_{211,i}\\ \bar A_{212,i}\end{bmatrix}, \bar E_{2,i}=\begin{bmatrix}\bar E_{21,i}\\ \bar E_{22,i}\end{bmatrix}, \quad (42)$$

where $\bar A_{211,i}\in R^{(p-(q+h))\times((n+h)-p)}$ and $\bar E_{22,i}\in\mathbb{R}^{(q+h)\times l}$. Then, $\tilde A_{11,i}$ from (40) can be written as

$$\tilde A_{11,i}=\bar A_{11,i}+\bar L_{1,i}\bar A_{211,i}. \quad (43)$$

By construction, $\bar L_{1,i}$ has been designed such that $\tilde A_{11,i}$ is quadratically stable.

For the system (37)-(39), consider a dynamical system

$$\dot{\hat z}_1=\sum_{i=1}^{k}\mu_i\left(\tilde A_{11,i}\hat z_1+\tilde A_{12,i}\hat z_2+\tilde B_{1,i}u-\tilde A_{12,i}e_y\right), \quad (44)$$

$$\dot{\hat z}_2=\sum_{i=1}^{k}\mu_i\left(\tilde A_{21,i}\hat z_1+\tilde A_{22,i}\hat z_2+\tilde B_{2,i}u+(\tilde A_{22,i}-\tilde A_{22}^{s})e_y+\upsilon\right), \quad (45)$$

$$\hat y=\hat z_2, \quad (46)$$

where $\tilde A_{22}^{s}$ is stable and υ is

$$\upsilon:=\begin{cases}(\hat\rho_{as}+\rho_0)\left\|\bar C_2\bar M_2\right\|_{\max}\dfrac{\tilde P_0 e_y}{\left\|\tilde P_0 e_y\right\|}, & e_y\neq 0\\ 0, & e_y(t)=0\end{cases}, \quad (47)$$

where $\left\|\bar C_2\bar M_2\right\|_{\max}$ is the maximal of each matrix norm $\left\|\bar C_2\bar M_{2,i}\right\|$ for i=1,...,k. The scalar ρ_0 will be described formally later. The matrix $\tilde P_0\in\mathbb{R}^{p\times p}$ verified the equation

$$\tilde P_0\tilde A_{22}^{s}+\tilde A_{22}^{sT}\tilde P_0=-Q_0, \quad (48)$$

where $Q_0\in\mathbb{R}^{p\times p}$. The function scalar $\hat\rho_{as}$ is

$$\frac{d\hat\rho_{as}}{dt}=\sigma\left\|\bar C_2\bar M_2\right\|_{\max}\left\|\tilde P_0 e_y\right\|, \sigma>0, \quad (49)$$

Let $e_1=z_1-\hat z_1$ and $e_y=z_2-\hat z_2=\bar y-\hat y$. Then from (37)-(39) and (44)-(46), we obtain

$$\dot e_1=\sum_{i=1}^{k}\mu_i\left(\tilde A_{11,i}e_1+\tilde E_{1,i}d\right), \quad (50)$$

$$\dot e_y=\sum_{i=1}^{k}\mu_i\left(\tilde A_{21,i}e_1+\tilde A_{22}^{s}e_y+\tilde M_{2,i}f+\tilde E_{2,i}d-\upsilon\right). \quad (51)$$

For the system (50)-(51), consider a sliding surface

$$S=\left\{\left(e_1(t),e_y(t)\right)\middle|e_y(t)=0\right\}. \quad (52)$$

Define r as

$$r=H\begin{pmatrix}e_1\\ e_y\end{pmatrix}, \quad (53)$$

where H is

$$H := \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}, \tag{54}$$

Let

$$\|\mathcal{H}\|_\infty := \sup_{\|d\|_2 \neq 0} \frac{\|r\|_2^2}{\|d\|_2^2}. \tag{55}$$

In the following, we present Theorem 1 which establishes the sufficient conditions for the existence of the proposed fuzzy SMO in terms of LMIs with a prescribed H∞ performance.

Stability Analysis

Theorem 1. The observer error dynamic system (50)-(51) is asymptotically stable satisfying the prescribed H∞ performance (55), if there exist matrices $\tilde P_1 = \tilde P_1^T > 0$, $\tilde P_0 = \tilde P_0^T > 0$ and $\bar W_i$ such that:

Minimize γ_1 subject to

$$\Omega_i = \begin{bmatrix} \Pi_1 & \tilde P_0 \bar C_2 \bar A_{21,i} & \tilde P_1 \bar E_{1,i} + \bar W_i \bar E_{22,i} \\ * & \Pi_2 & \tilde P_0 \bar C_2 \bar E_{2,i} \\ * & * & -\gamma_1 I \end{bmatrix} < 0, \tag{56}$$

where

$$\Pi_1 = \tilde P_1 \bar A_{11,i} + \bar A_{11,i}^T \tilde P_1 + \bar W_i \bar A_{211,i} + \bar A_{211,i}^T \bar W_i^T + H_1^T H_1,$$

$$\Pi_2 = \tilde A_{22}^T \tilde P_0 + \tilde P_0 \tilde A_{22} + H_2^T H_2,$$

and

$$\bar L_{1,i} = \tilde P_1^{-1} \bar W_i. \tag{57}$$

Proof.

Let

$$V = V_1 + V_2 + V_3, \tag{58}$$

where $V_1 = e_1^T \tilde P_1 e_1$, $V_2 = e_y^T \tilde P_0 e_y$ and $V_3 = \frac{1}{\sigma}\tilde\rho_{as}^2$ with $\tilde P_1 = \tilde P_1^T > 0$, $\tilde P_0 = \tilde P_0^T > 0$ are yet to be determined and $\tilde\rho_{as} = \rho_{as} - \hat\rho_{as}$. Its derivative along (50) is

$$\begin{aligned}
\dot V_1 &= \dot e_1^T \tilde P_1 e_1 + e_1^T \tilde P_1 \dot e_1 \\
&= \sum_{i=1}^{k} \mu_i \left(e_1^T \left(\tilde A_{11,i}^T \tilde P_1 + \tilde P_1 \tilde A_{11,i} \right) e_1 + 2 e_1^T \tilde P_1 \tilde E_{1,i} d \right) \\
&= \sum_{i=1}^{k} \mu_i \left(e_1^T \left(\left(\bar A_{11,i} + \bar L_{1,i} \bar A_{211,i} \right)^T \tilde P_1 + \tilde P_1 \left(\bar A_{11,i} + \bar L_{1,i} \bar A_{211,i} \right) \right) e_1 \right. \\
&\quad \left. + 2 e_1^T \tilde P_1 \left(\bar E_{1,i} + \bar L_{1,i} \bar E_{22,i} \right) d \right)
\end{aligned} \tag{59}$$

Similarly

$$\begin{aligned}
\dot V_2 &= \sum_{i=1}^{k} \mu_i \left(e_y^T (\tilde A_{22}^T \tilde P_0 + \tilde P_0 \tilde A_{22}) e_y + 2 e_y^T \tilde P_0 \tilde A_{21,i} e_1 \right. \\
&\quad \left. + 2 e_y^T \tilde P_0 \tilde E_{2,i} d + 2 e_y^T \tilde P_0 \tilde M_{2,i} f - 2 e_y^T \tilde P_0 \upsilon \right) \\
&= \sum_{i=1}^{k} \mu_i \left(e_y^T (\tilde A_{22}^T \tilde P_0 + \tilde P_0 \tilde A_{22}) e_y + 2 e_y^T \tilde P_0 \bar C_2 \bar A_{21,i} e_1 \right. \\
&\quad \left. + 2 e_y^T \tilde P_0 \bar C_2 \bar E_{2,i} d + 2 e_y^T \tilde P_0 \bar C_2 \bar M_{2,i} f - 2 e_y^T \tilde P_0 \upsilon \right).
\end{aligned} \tag{60}$$

Moreover, the time derivatives of V3 is

$$\dot V_3 = \frac{2}{\sigma} \tilde\rho_{as} \left(-\frac{d\hat\rho_{as}}{dt} \right). \tag{61}$$

From (59), (60) and (61), we have

$$\begin{aligned}
\dot V &= \sum_{i=1}^{k} \mu_i \left(e_1^T \left(\left(\bar A_{11,i} + \bar L_{1,i} \bar A_{211,i} \right)^T \tilde P_1 \right. \right. \\
&\quad \left. + \tilde P_1 \left(\bar A_{11,i} + \bar L_{1,i} \bar A_{211,i} \right) \right) e_1 + 2 e_1^T \tilde P_1 \left(\bar E_{1,i} + \bar L_{1,i} \bar E_{22,i} \right) d \\
&\quad + e_y^T (\tilde A_{22}^T \tilde P_0 + \tilde P_0 \tilde A_{22}) e_y + 2 e_y^T \tilde P_0 \bar C_2 \bar A_{21,i} e_1 \\
&\quad + 2 e_y^T (t) \tilde P_0 \tilde E_{2,i} d + 2 e_y^T \tilde P_0 \bar C_2 \bar M_{2,i} f \\
&\quad \left. - 2 e_y^T \tilde P_0 \upsilon + \frac{2}{\sigma} \tilde\rho_{as} \left(-\frac{d\hat\rho_{as}}{dt} \right) \right).
\end{aligned} \tag{62}$$

We have

$$\begin{aligned}
\sum_{i=1}^{k} \mu_i \left(e_y^T \tilde P_0 \bar C_2 \bar M_{2,i} f \right) &\le \sum_{i=1}^{k} \mu_i \left(\|\bar C_2 \bar M_{2,i}\| \|\tilde P_0 e_y\| \|f\| \right) \\
&\le \|\tilde P_0 e_y\| \|\bar C_2 \bar M_2\|_{\max} \|f\|
\end{aligned}, \tag{63}$$

with $\sum_{i=1}^{k} \mu_i \|\bar C_2 \bar M_{2,i}\| \le \|\bar C_2 \bar M_2\|_{\max}$.

Using (63), the last three terms of (62) can be written as

$$\begin{aligned}
&\sum_{i=1}^{k} \mu_i \left(2 e_y^T \tilde P_0 \bar C_2 \bar M_{2,i} f \right) - 2 e_y^T \tilde P_0 \upsilon + \frac{2}{\sigma} \tilde\rho_{as} \left(-\frac{d\hat\rho_{as}}{dt} \right) \\
&\le 2 \|\tilde P_0 e_y\| \|\bar C_2 \bar M_2\|_{\max} \|f\| - 2 \|\bar C_2 \bar M_2\|_{\max} (\hat\rho_{as} + \rho_0) \\
&\quad \times \frac{e_y^T \tilde P_0 \tilde P_0 e_y}{\|\tilde P_0 e_y\|} + \frac{2}{\sigma} \tilde\rho_{as} \left(-\frac{d\hat\rho_{as}}{dt} \right) \\
&= -2 \rho_0 \|\tilde P_0 e_y\| \|\bar C_2 \bar M_2\|_{\max} < 0
\end{aligned}. \tag{64}$$

From (64) it yields

$$\begin{aligned}
\dot V &\le \sum_{i=1}^{k} \mu_i \left(e_1^T \left(\left(\bar A_{11,i} + \bar L_{1,i} \bar A_{211,i} \right)^T \tilde P_1 \right. \right. \\
&\quad \left. + \tilde P_1 \left(\bar A_{11,i} + \bar L_{1,i} \bar A_{211,i} \right) \right) e_1 + 2 e_1^T \tilde P_1 \left(\bar E_{1,i} + \bar L_{1,i} \bar E_{22,i} \right) d \\
&\quad + e_y^T (\tilde A_{22}^T \tilde P_0 + \tilde P_0 \tilde A_{22}) e_y + e_y^T (\tilde A_{22}^T \tilde P_0 + \tilde P_0 \tilde A_{22}) e_y \\
&\quad \left. + 2 e_y^T \tilde P_0 \bar C_2 \bar A_{21,i} e_1 + 2 e_y^T \tilde P_0 \bar C_2 \bar E_{2,i} d \right).
\end{aligned} \tag{65}$$

Let

$$J_1 = \dot{V} + r^T r - \gamma_1 d^T d \quad (66)$$

From (65) and (66) we can get

$$\begin{aligned} J_1 &= \dot{V} + e_1^T H_1^T H_1 e_1 + e_y^T H_2^T H_2 e_y - \gamma_1 d^T d \\ &\le \sum_{i=1}^{k} \mu_i \Big(e_1^T \left(\left(\bar{A}_{11,i} + \bar{L}_{1,i}\bar{A}_{211,i} \right)^T \tilde{P}_1 \right. \\ &\quad + \tilde{P}_1\left(\bar{A}_{11,i} + \bar{L}_{1,i}\bar{A}_{211,i} \right) e_1 + 2e_1^T \tilde{P}_1\left(\bar{E}_{1,i} + \bar{L}_{1,i}\bar{E}_{22,i} \right) d \\ &\quad + e_y^T (\tilde{A}_{22}^T \tilde{P}_0 + \tilde{P}_0\tilde{A}_{22}) e_y + 2e_y^T \tilde{P}_0\bar{C}_2\bar{A}_{21,i} e_1 \\ &\quad + 2e_y^T \tilde{P}_0\bar{C}_2\bar{E}_{2,i} d + e_1^T H_1^T H_1 e_1 + e_y^T H_2^T H_2 e_y \\ &\quad - \gamma_1 d^T d \Big) \\ &= \zeta^T \Omega \zeta \end{aligned} , \quad (67)$$

where

$$\Omega = \sum_{i=1}^{k} \mu_i \begin{bmatrix} \Pi_1 & \tilde{P}_0\bar{C}_2\bar{A}_{21,i} & \tilde{P}_1\left(\bar{E}_{1,i} + \bar{L}_{1,i}\bar{E}_{22,i} \right) \\ * & \Pi_2 & \tilde{P}_0\bar{C}_2\bar{E}_{2,i} \\ * & * & -\gamma_1 I \end{bmatrix} \quad (68)$$

and

$$\zeta = \begin{bmatrix} e_1 \\ e_y \\ d \end{bmatrix},$$

$$\Pi_1 = \left(\bar{A}_{11,i} + \bar{L}_{1,i}\bar{A}_{211,i} \right)^T \tilde{P}_1 + \tilde{P}_1\left(\bar{A}_{11,i} + \bar{L}_{1,i}\bar{A}_{211,i} \right) + H_1^T H_1,$$

$$\Pi_2 = \tilde{A}_{22}^T \tilde{P}_0 + \tilde{P}_0\tilde{A}_{22} + H_2^T H_2.$$

Obviously, $\Omega < 0$ means $J_1 < 0$. Apply the Schur complement, $\Omega < 0$ is equivalent to

$$\sum_{i=1}^{k} \mu_i \Omega_i < 0 . \quad (69)$$

We can find that the form of the matrix Ω_i is nonlinear. Denoting $\bar{W}_i = \tilde{P}_1\bar{L}_{1,i}$, we can obtain the LMIs form (56).

Under the zero initial condition, we obtain

$$\begin{aligned} \int_0^\infty \left[\|r\|^2 - \gamma_1 \|d\|^2 \right] dt &= \int_0^\infty \left[\|r\|^2 - \gamma_1 \|d\|^2 + \dot{V} \right] dt - \int_0^\infty \dot{V} dt \\ &= \int_0^\infty \left[\|r\|^2 - \gamma_1 \|d\|^2 + \dot{V} \right] dt - V(\infty) \\ &\quad + V(0) \\ &= \int_0^T V_0 dt < 0 \end{aligned} . \quad (70)$$

Thus $J_1 < 0$ implies

$$\int_0^T \left(r^T r \right) dt \le \gamma_1 \int_0^T \left(d^T d \right) dt , \quad (71)$$

namely

$$\|r\|_2^2 \le \gamma_1 \|d\|_2^2 . \quad (72)$$

So if $\Omega_i < 0$, then the system (50)-(51) is asymptotically stable with disturbances attenuation $\sqrt{\gamma_1}$. □

Using the inverse transformation of $T_{L,i}T_i$, we have

$$\bar{G}_{l,i} = \left(T_{L,i}T_i \right)^{-1} \begin{bmatrix} \tilde{A}_{12,i} \\ \tilde{A}_{22,i} - \tilde{A}_{22} \end{bmatrix}$$

$$\bar{G}_{n,i} = \left(T_{L,i}T_i \right)^{-1} \begin{bmatrix} 0_{(n+h-p)\times p} \\ I_p \end{bmatrix} . \quad (73)$$

Sliding Motion Analysis

Theorem 2. The error dynamics system (50)-(51) is driven to the sliding surface S (52) in finite time and remain on it if the LMIs (56) are solvable and the gain ρ_0 satisfy

$$\rho_0 \ge \frac{\left\| \bar{C}_2\bar{A}_{21} \right\|_{\max} \varpi + \left\| \bar{C}_2\bar{E}_2 \right\|_{\max} \|d(t)\| + \rho_1}{\left\| \bar{C}_2\bar{M}_2 \right\|_{\max}} , \quad (74)$$

where ρ_1 is small positive constant.

Proof. Consider the Lyapunov function $V_s = \frac{1}{2}\left(e_y^T \tilde{P}_0 e_y + \frac{1}{\sigma}\tilde{\rho}_{as}^2 \right)$. The differentia of V_s along the error dynamic (51) is

$$\begin{aligned} \dot{V}_s &= \sum_{i=1}^{k} \mu_i \left(e_y^T \tilde{P}_0\left(\bar{C}_2\bar{A}_{21,i}e_1 + \tilde{A}_{22}e_y + \bar{C}_2\bar{M}_{2,i} f + \bar{C}_2\bar{E}_{2,i} - \upsilon \right) \right) \\ &\quad + \frac{1}{\sigma}\tilde{\rho}_{as}\left(-\frac{d\hat{\rho}_{as}}{dt} \right). \end{aligned} \quad (75)$$

Since $\tilde{A}_{22}$ is stable and $\tilde{P}_0$ is solution to the Lyapunov equation (48), the following relation will be shown

$$e_y^T(t)\tilde{P}_0\tilde{A}_{22}e_y(t) = -\frac{1}{2}e_y^T(t)Q_0 e_y(t) \le 0 . \quad (76)$$

Using relation (76), it follows

$$\begin{aligned} \dot{V}_s &= \sum_{i=1}^{k} \mu_i \left(e_y^T \tilde{P}_0\left(\bar{C}_2\bar{A}_{21,i}e_1 + \bar{C}_2\bar{M}_{2,i} f + \bar{C}_2\bar{E}_{2,i} d - \upsilon \right) \right) \\ &\quad + \frac{1}{\sigma}\tilde{\rho}_{as}\left(-\frac{d\hat{\rho}_{as}}{dt} \right). \end{aligned} \quad (77)$$

Using Assumption 1 and the argument above $\|e_1\| \le \varpi$, we can obtain

$$e_y^T \tilde{P}_0 \bar{C}_2 \bar{A}_{21,i} e_1 \le \left\| \bar{C}_2 \bar{A}_{21,i} \right\| \left\| \tilde{P}_0 e_y \right\| \varpi, \tag{78}$$

$$e_y^T \tilde{P}_0 \bar{C}_2 \bar{M}_{2,i} f \le 2 \left\| \bar{C}_2 \bar{M}_{2,i} \right\| \left\| \tilde{P}_0 e_y \right\| \rho_{as}, \tag{79}$$

$$e_y^T \tilde{P}_0 \bar{C}_2 \bar{E}_{2,i} d \le 2 \left\| \bar{C}_2 \bar{E}_{2,i} \right\| \left\| \tilde{P}_0 e_y \right\| \|d\|. \tag{80}$$

Applying relations (78)-(80), we have

$$\dot{V}_s \le \sum_{i=1}^{k} \mu_i \Big(\left\| \tilde{P}_0 e_y \right\| \Big(\left\| \bar{C}_2 \bar{A}_{21,i} \right\| \varpi + \left\| \bar{C}_2 \bar{M}_{2,i} \right\| \rho_{as} + \left\| \bar{C}_2 \bar{E}_{2,i} \right\| \|d\| \Big) \Big) - \upsilon + \frac{1}{\sigma} \tilde{\rho}_{as} \Big(-\frac{d\hat{\rho}_{as}}{dt}\Big). \tag{81}$$

Since

$$\sum_{i=1}^{k} \mu_i \left(\left\| \bar{C}_2 \bar{A}_{21,i} \right\| \varpi + \left\| \bar{C}_2 \bar{F}_{2,i} \right\| \rho_{as} + \left\| \bar{C}_2 \bar{E}_{2,i} \right\| \|d\| \right) \le \left\| \bar{C}_2 \bar{A}_{21} \right\|_{\max} \varpi + \left\| \bar{C}_2 \bar{M}_2 \right\|_{\max} \rho_{as} + \left\| \bar{C}_2 \bar{E}_2 \right\|_{\max} \|d\|. \tag{82}$$

It thus follows from (82) that

$$\begin{aligned} \dot{V}_s &\le \left\| \tilde{P}_0 e_y \right\| \Big(\left\| \bar{C}_2 \bar{A}_{21} \right\|_{\max} \varpi + \left\| \bar{C}_2 \bar{M}_2 \right\|_{\max} \rho_{as} + \left\| \bar{C}_2 \bar{E}_2 \right\|_{\max} \|d\| \Big) - (\hat{\rho}_{as} + \rho_0) \left\| \bar{C}_2 \bar{M}_2 \right\|_{\max} \\ &\quad \times \frac{e_y^T \tilde{P}_0 \tilde{P}_0 e_y}{\left\| \tilde{P}_0 e_y \right\|} + \frac{1}{\sigma} \tilde{\rho}_{as} \Big(-\frac{d\hat{\rho}_{as}}{dt}\Big) \\ &= \left\| \tilde{P}_0 e_y \right\| \Big(\left\| \bar{C}_2 \bar{A}_{21} \right\|_{\max} \varpi + \left\| \bar{C}_2 \bar{E}_2 \right\|_{\max} \|d\| - \rho_0 \left\| \bar{C}_2 \bar{M}_2 \right\|_{\max} \Big). \end{aligned} \tag{83}$$

If

$$\dot{V}_s \le -\rho_1 \left\| \tilde{P}_0 e_y \right\| \le -\rho_1 \sqrt{\lambda_{\min}(\tilde{P}_0)} V_s^{1/2} \tag{84}$$

Then the reachability condition [15] is verified. □

FAULT ESTIMATION

From Theorem 2, an ideal sliding mode take place on S and $e_y = \dot{e}_y = 0$. During the sliding motion, (51) becomes:

$$0 = \sum_{i=1}^{k} \mu_i \left(\bar{C}_2 \bar{A}_{21,i} e_1 + \bar{C}_2 \bar{M}_{2,i} f + \bar{C}_2 \bar{E}_{2,i} d - \upsilon_{eq} \right), \tag{85}$$

where υ_{eq} denotes the equivalent term [16] replaced by

$$\upsilon_{eq} := (\hat{\rho}_{as} + \rho_0) \left\| \bar{C}_2 \bar{M}_2 \right\|_{\max} \frac{\tilde{P}_0 e_y}{\left\| \tilde{P}_0 e_y \right\| + \delta}, \delta > 0 \tag{86}$$

Since C_2 is invertible, (85) can be rewritten as

$$\sum_{i=1}^{k} \mu_i \left(\bar{C}_2^{-1} \upsilon_{eq} - \bar{M}_{2,i} f \right) = \sum_{i=1}^{k} \mu_i \left(\bar{A}_{21,i} e_1 + \bar{E}_{2,i} d \right) \tag{87}$$

Considering $\|r\|_2^2 \le \gamma_1 \|d\|_2^2$ for some $\gamma_1 > 0$ and $r = He$, $e = \begin{pmatrix} e_1^T & e_y^T \end{pmatrix}^T$, it follows that

$$\begin{aligned} \sum_{i=1}^{k} \mu_i \left(\bar{A}_{21,i} e_1 + \bar{E}_{2,i} d \right) &\le \sum_{i=1}^{k} \mu_i \left(\left\| \bar{A}_{21,i} \right\| \|e_1\| + \left\| \bar{E}_{2,i} \right\| \|d\| \right) \\ &\le \sqrt{\gamma_1} \left\| \bar{A}_{21} \right\|_{\max} \sigma_{\max}(H^{-1}) \|d\| + \left\| \bar{E}_2 \right\|_{\max} \|d\| \\ &= \left(\sqrt{\gamma_1} \beta_1 + \beta_2 \right) \|d\|, \end{aligned} \tag{88}$$

where $\beta_1 = \left\| \bar{A}_{21} \right\|_{\max} \sigma_{\max}(H^{-1})$ and $\beta_2 = \left\| \bar{E}_2 \right\|_{\max}$. Thus for a small $\left(\sqrt{\gamma_1} \beta_1 + \beta_2 \right) \|d(t)\|$, we obtain

$$\bar{C}_2^{-1} \upsilon_{eq} \approx \sum_{i=1}^{k} \mu_i \bar{M}_{2,i} f \tag{89}$$

The faults f_a and f_s can be then approximated as

$$\hat{f} = \begin{bmatrix} \hat{f}_a^T & \hat{f}_s^T \end{bmatrix}^T \approx \left(\sum_{i=1}^{k} \mu_i \bar{M}_{2,i} \right)^+ \bar{C}_2^{-1} (\hat{\rho}_{as} + \rho_0) \times \left\| \bar{C}_2 \bar{M}_2 \right\|_{\max} \frac{\tilde{P}_0 e_y}{\left\| \tilde{P}_0 e_y \right\| + \delta} \tag{90}$$

FAULT TOLERANT CONTROL DESIGN

Define corrected output as

$$y_c = Cx + N\left(f_s - \hat{f}_s \right) \tag{91}$$

System (3)-(4) becomes

$$\dot{x} = \sum_{i=1}^{k} \mu_i \left(A_i x + B_i u + M_i f_a + E_i d \right), \tag{92}$$

$$y_c = Cx + N e_{fs} \tag{93}$$

where $e_{f_s} = f_s - \hat{f}_s$ is the sensor fault estimation error. A static output feedback fault-tolerant control (SOFFTC) law [17] is designed as follows

$$u = \sum_{i=1}^{k} \mu_i \left(K_i y_c - G_i \hat{f}_a \right), \tag{94}$$

where K_i and G_i are gains matrices to be determined. Substituting (94) in (92), we have

$$\dot{x} = \sum_{i=1}^{k} \sum_{j=1}^{k} \mu_i \mu_j \big(A_i x + B_i (K_j C x + K_j N e_{fs} - G_j \hat{f}_a) + M_i f_a + E_i d \big). \tag{95}$$

The gain G_i is designed so that $B_iG_i = M_i$ [18] where B_i^+ is the pseudo inverse of B_i.,It follows that

$$\dot{x} = \sum_{i=1}^{k}\sum_{j=1}^{k}\mu_i\mu_j\Big(\big(A_i + B_iK_jC\big)x + B_iK_jNe_{fs} + M_ie_{fa} + E_id\Big) \tag{96}$$

where $e_{fa} = f_a - \hat{f}_a$. Then, we get

$$\dot{x} = \sum_{i=1}^{k}\sum_{j=1}^{k}\mu_i\mu_j\Big(\big(A_i + B_iK_jC\big)x + \bar{B}_{ij}\varphi\Big), \tag{97}$$

$$y_c = Cx + Ne_{fs}, \tag{98}$$

where $\bar{B}_{ij} = \begin{bmatrix} B_iK_jN & M_i & E_i \end{bmatrix}$ and $\varphi = \begin{bmatrix} e_{fs}^T & e_{fa}^T & d^T \end{bmatrix}^T$.

Firstly, the following lemmas will be used

Lemma 3. For matrices A and B, we have

$$AB + (AB)^T \le \varepsilon^{-1}AA^T + \varepsilon B^TB, \quad \varepsilon > 0. \tag{99}$$

Lemma 4. If

$$S_{ii} < 0, \quad 1 \le i \le k, \tag{100}$$

$$\frac{2}{k-1}S_{ii} + S_{ij} + S_{ji} < 0, \quad 1 \le i \ne j \le k, \tag{101}$$

we have

$$\sum_{i=1}^{k}\sum_{j=1}^{k}\mu_i\mu_jS_{ij} < 0. \tag{102}$$

Theorem 3. The closed-loop system (97)-(98) is robust stable with H∞ performance index

$$\|y_c\|_2^2 < \gamma_c\|\varphi\|_2^2, \tag{103}$$

if there exist $Q = Q^T > 0$, R and S_j, such as:

Minimize γ_c subject to

$$\Xi_{ii} < 0,\ i = 1,\dots,k, \tag{104}$$

$$\frac{2}{k-1}\Xi_{ii} + \Xi_{ij} + \Xi_{ji} < 0,\ 1 \le i \ne j \le k, \tag{105}$$

$$CQ = RC, \tag{106}$$

where

$$\Xi_{ij} = \begin{bmatrix} \Psi_{1,ij} & 0 & M_i & N_i & QC^T & B_iK_j & 0 \\ * & \Psi_{2,ij} & 0 & 0 & QN^T & 0 & QC \\ * & * & -\gamma_cI & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma_cI & 0 & 0 & 0 \\ * & * & * & * & -\gamma_cI & 0 & 0 \\ * & * & * & * & * & -\varepsilon^{-1}I & 0 \\ * & * & * & * & * & * & -\varepsilon I \end{bmatrix},$$

with

$$\bar{\Psi}_{1,ij} = A_iQ + QA_i + B_iS_jC + C^TS_j^TB_i^T, \tag{107}$$

$$\bar{\Psi}_{2,ij} = -2\gamma_cQ + \gamma_cI. \tag{108}$$

Furthermore, the gains of SOFFTC are given by $K_i = S_iR^{-1}$.

Proof. Choose $V_x = x^TPx$, where $P = P^T > 0$. Its derivative is

$$\dot{V}_x = \sum_{i=1}^{k}\sum_{j=1}^{k}\mu_i\mu_j\Big(\Big(x^T\big((A_i + B_iK_jC)^TP + P(A_i + B_iK_jC)\big)x\Big) + 2x^TP\bar{B}_{ij}\varphi\Big). \tag{109}$$

Let

$$J_2 = \dot{V}_x + \frac{1}{\gamma_c}y_c^Ty_c - \gamma_c\varphi^T\varphi, \tag{110}$$

where

$$y_c^Ty_c = x^TC^TCx + x^TC^TNe_{fs} + e_{fs}^TN^TCx + e_{fs}^TN^TNe_{fs}. \tag{111}$$

Define $Z = \begin{bmatrix} N & 0 & 0 \end{bmatrix}$, then

$$y_c^Ty_c = x^TC^TCx + \varphi^TZ^TZ\varphi + 2x^TC^TZ\varphi. \tag{112}$$

So we can get that

$$\begin{aligned} J_2 &= \sum_{i=1}^{k}\sum_{j=1}^{k}\mu_i\mu_j\Big(x^T\big((A_i + B_iK_jC)^TP + P(A_i + B_iK_jC)\big)x \\ &\quad + 2x^TP\bar{B}_{ij}\varphi\Big) + \frac{1}{\gamma_c}x^TC^TCx + \frac{1}{\gamma_c}\varphi^TZ^TZ\varphi \\ &\quad + \frac{2}{\gamma_c}x^TC^TZ\varphi - \gamma_c\varphi^T\varphi \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k}\mu_i\mu_j\left(\begin{bmatrix} x \\ \varphi \end{bmatrix}^T\begin{bmatrix} \Upsilon_{ij} & P\bar{B}_{ij} + \frac{1}{\gamma_c}C^TZ \\ * & -\gamma_cI + \frac{1}{\gamma_c}Z^TZ \end{bmatrix}\begin{bmatrix} x \\ \varphi \end{bmatrix}\right). \end{aligned} \tag{113}$$

where

$$\Upsilon_{ij} = (A_i + B_i K_j C)^T P + P(A_i + B_i K_j C) + \frac{1}{\gamma_c} C^T C .$$

Thus, $J_2 < 0$, if

$$\sum_{i=1}^{k}\sum_{j=1}^{k}\mu_i\mu_j \begin{bmatrix} \Upsilon_{ij} & P\bar{B}_{ij} + \frac{1}{\gamma_c} C^T Z \\ * & -\gamma_c I + \frac{1}{\gamma_c} Z^T Z \end{bmatrix} < 0 . \quad (114)$$

By applying Schur complement, (114) can be written as

$$\sum_{i=1}^{k}\sum_{j=1}^{k}\mu_i\mu_j \begin{bmatrix} \bar{\Upsilon}_{ij} & PB_iK_jN & PM_i & PE_i & C^T \\ * & -\gamma_c I & 0 & 0 & N^T \\ * & * & -\gamma_c I & 0 & 0 \\ * & * & * & -\gamma_c I & 0 \\ * & * & * & * & -\gamma_c I \end{bmatrix} < 0 , \quad (115)$$

where

$$\bar{\Upsilon}_{ij} = (A_i + B_i K_j C)^T P + P(A_i + B_i K_j C) .$$

Premultiplying and postmultiplying by $X = diag\left\{P^{-1}, P^{-1}, I, I\right\}$ and its transpose in (115), then we obtain

$$\sum_{i=1}^{k}\sum_{j=1}^{k}\mu_i\mu_j \begin{bmatrix} \Psi_{1,ij} & B_iK_jNQ & M_i & N_i & QC^T \\ * & -\gamma_c QQ & 0 & 0 & QN^T \\ * & * & -\gamma_c I & 0 & 0 \\ * & * & * & -\gamma_c I & 0 \\ * & * & * & * & -\gamma_c I \end{bmatrix} < 0 , \quad (116)$$

where $\Psi_{1,ij} = A_i Q + QA_i + B_i K_j CQ + Q^T C^T K_j^T B_i^T$ and $Q = P^{-1}$. Based on Lemma 3, it is easy to obtain that

$$Q + Q \le QQ + I . \quad (117)$$

From γ_c, (117) is equivalent to

$$-\gamma_c QQ \le -2\gamma_c Q + \gamma_c I . \quad (118)$$

Thus, we can obtain

$$\sum_{i=1}^{k}\sum_{j=1}^{k}\mu_i\mu_j \Phi < 0 , \quad (119)$$

where

$$\Phi = \begin{bmatrix} \Psi_{1,ij} & B_iK_jNQ & M_i & N_i & QC^T \\ * & \Psi_{2,ij} & 0 & 0 & QN^T \\ * & * & -\gamma_c I & 0 & 0 \\ * & * & * & -\gamma_c I & 0 \\ * & * & * & * & -\gamma_c I \end{bmatrix} .$$

where $\Psi_{2,ij} = -2\gamma_c Q + \gamma_c I$.

Φ can be further decomposed as below:

$$\Sigma + HG + (HG)^T < 0 , \quad (120)$$

where

$$\Sigma = \begin{bmatrix} \Psi_{1,ij} & 0 & M_i & N_i & QC^T \\ * & \Psi_{2,ij} & 0 & 0 & QN^T \\ * & * & -\gamma_c I & 0 & 0 \\ * & * & * & -\gamma_c I & 0 \\ * & * & * & * & -\gamma_c I \end{bmatrix},$$

$$H = \begin{bmatrix} K_j^T B_i^T & 0 & 0 & 0 & 0 \end{bmatrix}^T ,$$

$$G = \begin{bmatrix} 0 & NQ & 0 & 0 & 0 \end{bmatrix}.$$

By using Lemma 3, it is follows that

$$\Sigma + HG + (HG)^T \le \Sigma + \varepsilon^{-1} HH^T + \varepsilon G^T G . \quad (121)$$

From (121), (119) is equivalent to

$$\sum_{i=1}^{k}\sum_{j=1}^{k}\mu_i\mu_j \Delta_{ij} < 0 , \quad (122)$$

where

$$\Delta_{ij} = \begin{bmatrix} \Psi_{1,ij} & 0 & M_i & N_i & QC^T & B_iK_j & 0 \\ * & \Psi_{2,ij} & 0 & 0 & QN^T & 0 & \varepsilon QC \\ * & * & -\gamma_c I & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma_c I & 0 & 0 & 0 \\ * & * & * & * & -\gamma_c I & 0 & 0 \\ * & * & * & * & * & -\varepsilon I & 0 \\ * & * & * & * & * & * & -\varepsilon I \end{bmatrix} .$$

Notice that the inequality (122) is not jointly convex in K_j and Q. Then, define $CQ = RC$ and $K_i R = S_i$, so that $K_j CQ = S_j C$. Substituting the result into (122) yields

$$\sum_{i=1}^{k}\sum_{j=1}^{k}\mu_i\mu_j \Xi_{ij} \le 0 , \quad (123)$$

where Ξ_{ij} are the same forms as that in (107). Then if (104)-(105) are verified, then (124) holds.

III. INVERTED PENDULUM EXAMPLE

In this example, an inverted pendulum on a cart is considered [21]. The dynamical equations are given by:

$$\begin{cases}\dot{x}_1 = x_2\\[4pt] \dot{x}_2 = \dfrac{g\sin(x_1) - mlax_2^2\dfrac{\sin(2x_1)}{2} - ba\cos(x_1)x_4 - a\cos(x_1)F}{\dfrac{4l}{3} - mla\cos(x_1)^2}\\[4pt] \dot{x}_3 = x_4\\[4pt] \dot{x}_4 = \dfrac{-mga\dfrac{\sin(2x_1)}{2} + \dfrac{4mla}{3}x_2^2\sin(x_1) - bax_4 + \dfrac{4a}{3}(F - f_c)}{\dfrac{4}{3} - ma\cos(x_1)^2}\end{cases}$$

where x_1 is the angular position, x_2 is the angular velocity, x_3 is the cart position, x_4 is the cart velocity, m is the pendulum mass, M is the cart mass, $g = 9.8m/s$ is the gravity constant, and $a = 1/(m+M)$. In all simulations, $m = 2.0kg$, $M = 0.8kg$, $l = 0.5m$, and $L = 2m$.

We consider that the nonlinear system is represented by two local model T-S fuzzy models $(k = 2)$ and is given by

$$\begin{cases}\dot{x} = \sum_{i=1}^{2}\mu_i\left(A_i x + B_i u + M_i f_a + E_i d\right)\\ y = Cx + Nf_s\end{cases},$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0\\ \dfrac{g}{\dfrac{4l}{3} - mla} & 0 & 0 & \dfrac{ba}{\dfrac{4l}{3} - mla}\\ 0 & 0 & 0 & 1\\ \dfrac{-mga}{\dfrac{4}{3} - ma} & 0 & 0 & \dfrac{-ba}{\dfrac{4}{3} - ma}\end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0\\ \dfrac{g\dfrac{2\sqrt{2}}{\pi}}{\dfrac{4}{3} - \dfrac{mla}{2}} & 0 & 0 & \dfrac{ba\dfrac{\sqrt{2}}{2}}{\dfrac{4l}{3} - \dfrac{mla}{2}}\\ 0 & 0 & 0 & 1\\ \dfrac{-mga\dfrac{2}{\pi}}{\dfrac{4}{3} - \dfrac{ma}{2}} & 0 & 0 & -\dfrac{ba}{\dfrac{4}{3} - \dfrac{ma}{2}}\end{bmatrix}$$

$$B_1 = M_1 = \begin{bmatrix} 0\\ \dfrac{-a}{\dfrac{4l}{3} - mla}\\ 0\\ \dfrac{\dfrac{4a}{3}}{\dfrac{4}{3} - ma}\end{bmatrix}, B_2 = M_2 = \begin{bmatrix} 0\\ \dfrac{-a\dfrac{\sqrt{2}}{2}}{\dfrac{4l}{3} - \dfrac{mla}{2}}\\ 0\\ \dfrac{\dfrac{4a}{3}}{\dfrac{4}{3} - \dfrac{ma}{2}}\end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}, N = \begin{bmatrix} 0\\ 1\\ 0\end{bmatrix}$$

Assume that the fuzzy rules as follows:

$$\mu_1(x_1) = \frac{1 - \dfrac{1}{1 + \exp(-14(x_1 - \frac{\pi}{8}))}}{1 + \exp(-14(x_1 + \frac{\pi}{8}))},$$

$$\mu_2(x_1) = 1 - \mu_1(x_1)$$

Fault estimation design. Choose $A_f = 1$, $\tilde{A}_{22} = diag(-3, -5, -7)$, $H_1 = I_{2\times 2}$, $H_2 = I_{3\times 3}$, $\sigma = 0.1$, $\delta = 0.02$ and $\rho_0 = 10$. By solving Theorem 1, one obtains $\gamma_1 = 0.2734$ with

$$\bar{L}_{1,1} = \begin{bmatrix} 7.1431\\ -13.5658\end{bmatrix}, \bar{L}_{1,2} = \begin{bmatrix} 7.1053\\ -9.1142\end{bmatrix},$$

and the fuzzy sliding mode observer gain matrices

$$\bar{G}_{l,1} = \begin{bmatrix} -1.5000 & -18.5658 & 0\\ -4.4118 & -85.5867 & 0\\ 1 & 20.2038 & 0\\ 2.9559 & -23.3007 & 0\\ 0 & -5.0509 & 6\end{bmatrix}$$

$$\bar{G}_{n,1} = \begin{bmatrix} 0 & -1 & 0\\ -1.5000 & -13.5658 & 0\\ 0 & 5.0509 & 0\\ 1 & -5.0509 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\bar{G}_{l,2} = \begin{bmatrix} -1.0607 & -14.1142 & 0\\ -3.1246 & -60.1815 & 0\\ 1 & 20.0968 & 0\\ 2.9595 & -23.9046 & 0\\ 0 & -5.0242 & 6\end{bmatrix}$$

$$\bar{G}_{n,2} = \begin{bmatrix} 0 & -1 & 0\\ -1.0607 & -9.1142 & 0\\ 0 & 5.0242 & 0\\ 1 & -5.0242 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Fault tolerant controller. By solving Theorem 3, we obtain $\gamma_c = 3.1005$ and the controller gain matrices

$$K_1 = \begin{bmatrix} -2.7838 & -6.8067 & -11.1030\end{bmatrix}$$

$$K_2 = \begin{bmatrix} -2.1520 & -7.5055 & -11.8265\end{bmatrix}$$

Simulation results. Assume the actuator fault f_a and sensor fault f_s are given as

$$f_a(t) = \begin{cases} 0 & t < 2\\ 0.5\sin(\pi(t-2)) & t \geq 2\end{cases}$$

$$f_s(t) = \begin{cases} 0 & t < 5\\ 1 & t \geq 5\end{cases}$$

Fig. 1 presents actuator fault-estimation simulation results. Fig. 2 shows the simulation result of sensor fault estimation.

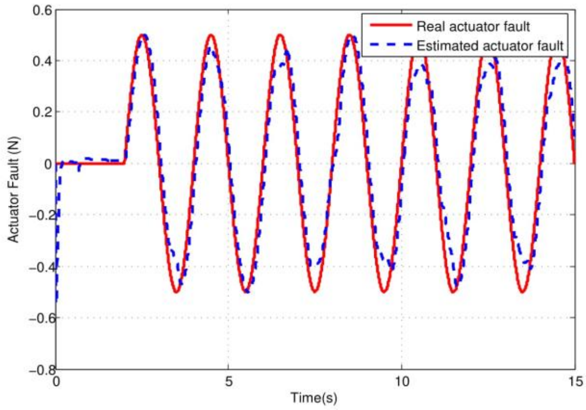

Fig. 1.Actuator fault f_a and its estimate $\hat{f}_a$.

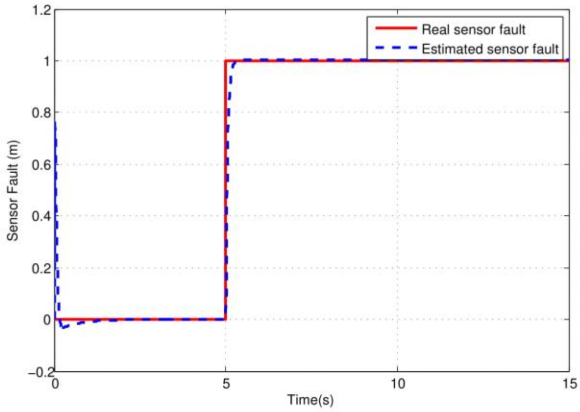

Fig. 2.Sensor fault f_s and its estimate $\hat{f}_s$.

Fig. 3 illustrate system output response y_2. We can observe that the proposed SMO design achieves the desired performance and the SOFFTC guarantee the stability of the closed-loop system.

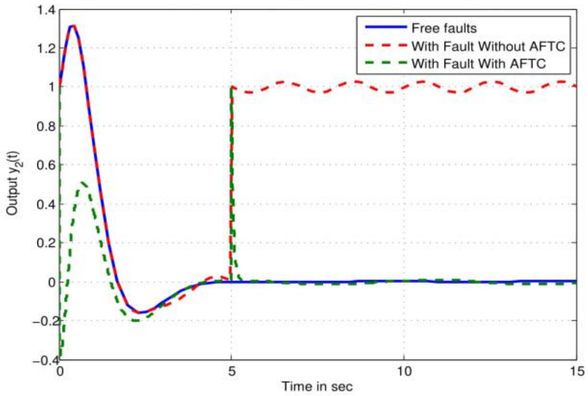

Fig. 3.Output response y_2

IV. CONCLUSION

In this paper, robust fault estimation and fault-tolerant control method has been designed for T-S fuzzy systems in presence of simultaneous actuator faults, sensor faults and external disturbances. Firstly, using H∞ technique, an adaptive fuzzy sliding mode observer SMO has been developed to estimate system state and faults signals. Then, a SOFFTC has been designed to stabilize the closed-loop system. Finally, a non-linear model of inverted pendulum with cart has been used to show the efficiency of the proposed observer and controller.

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